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The Wave Packet Formalism of Neutrino Oscillations and the Maximal Mixing Scenario

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Some aspects of the neutrino oscillation phenomena are studied in this contribution in the framework of quantum mechanics wave packet formalism. Based on the experimental arguments we consider the implications of the threefold maximal mixing scenarios in this formalism. Some possible predicted consequences are discussed.

Keywords: neutrino oscillations, wave packet formalism, maximal mixing

1. Introduction

Neutrinos can oscillate if neutrino mixing is realised in nature, i.e. if the left-handed components (L) of the flavour neutrino fields $(\alpha = e,\ \mu,\ \tau)$ are superposition of the left-handed components of the fields of neutrinos with definite mass (a):

$$\nu_{\alpha L} = \sum_a U_{\alpha a} \nu_{aL} .$$

In this equation U is the neutrino mixing matrix.

Neutrino oscillations are possible only if the processes of neutrino production and detection are coherently in the Heisenberg uncertainty relation sense. The quantum mechanics of neutrino oscillations have been studied in several papers, see for example [1] and reviewed in [2]. Alternative derivations of neutrino oscillations in the framework of quantum field theory also exist.

The standard theory of neutrino oscillations in vacuum has been developed on the basis of following main assumptions:

- Neutrinos are extremely relativistic particles.
- They are produced in charged-current weak interaction processes. For neutrino state $|\nu_k\rangle$ with mass m_k, their energy and momentum are connected by the relativistic dispersion relation $E_k^2 = p_k^2 + m_k^2$. Because in oscillation experiments neutrinos propagate along a macroscopic distance between production and detection, only one spatial direction along the neutrino path is considered.
- The approximation $t \cong L$ is used. This approximation is justified because in real experiments, the propagation time is difficult or impossible to be measured, but currently the source-- detector distance L is known.
- The massive neutrino states have the same momentum ("equal momentum assumption"), but different energies, or alternatively, the same energy, but different momenta. From these different hypotheses, different oscillation probabilities and consequently controversial claims could be found in the literature.

A simple quantum mechanical treatment of neutrinos as plane waves is currently used. The massive neutrino states $|\nu_k\rangle$ evolve in time according to the Schrodinger equation, whose solution is

$$|\nu_k(t)\rangle = e^{-iE_k t} |\nu_k\rangle .$$

The oscillation probability for the process between the states $\nu_\alpha \leftrightarrow \nu_\beta$ is:

$$P_{\alpha\beta}(L) = \sum_a |U_{\alpha a}|^2 |U_{\beta a}|^2 + 2\,\mathrm{Re} \sum_{a>b} \left| U_{\alpha a}^* U_{\beta a} U_{\alpha b} U_{\beta b}^* \right| \cdot \exp\left(-i \frac{\Delta m_{ab}^2}{2} \cdot \frac{L}{E} \right)$$

as a function of the distance L, with $\Delta m_{ab}^2 \equiv m_a^2 - m_b^2$.

In this contribution, the phenomena of neutrino oscillation are studied using the rigorous treatment in the framework of quantum mechanics, utilising a wave packet formalism. Based on the experimental arguments, we consider the implications of the threefold maximal mixing scenario in this formalism. Some conclusions have been derived.

2. The wave packet formalism of neutrino oscillations

The wave packet formalism has been developed in 1981 by Boris Kayser [3].

In this formalism, a neutrino state is constructed with momentum distributed around a mean value of the $\vec{p}_a$, supposing a Gaussian form, where σ is the width of the distribution. The average momenta of the different mass eigenstates are determinates by the kinematics of the production process.

The transition probability is space is obtained from the average of

$$P_{\alpha\beta}(L,t) \propto \left|A_{\alpha\beta}(L,t)\right|^2$$

over the unmeasured propagation time. From the time integration of transition probability, in the relativistic approximation, on obtains:

$$P_{\alpha\beta}(L) = \sum_a |U_{\alpha a}|^2 |U_{\beta a}|^2 + 2\,\mathrm{Re}\sum_{a>b}\left|U^*_{\alpha a}U_{\beta a}U_{\alpha b}U^*_{\beta b}\right| \cdot \exp\left[-2\pi i\frac{L}{L^{osc}_{ab}} - \left(\frac{L}{L^{coh}_{ab}}\right)^2 - 2\pi^2\rho^2\omega\left(\frac{L}{L^{osc}_{ab}}\right)^2\right]$$

where

$L^{osc}_{ab} = \dfrac{4\pi E}{\left|\Delta m^2_{ab}\right|}$ are the oscillations lengths

and

$L^{coh}_{ab} = \dfrac{4\sqrt{2}E^2}{\left|\Delta m^2_{ab}\right|}\sigma_x$ are the coherence lengths.

The parameter $\rho^2\omega$ characterises the kinematics of the production and detection processes [4].

Some comments are necessary about the width of the neutrino packets. The initial neutrino state is a coherent superposition of mass eigenstates. This is possible because neutrino masses are small and the differences between the momenta of different mass eigenstates is smaller than the momentum uncertainty of the production process due to its localisation in space. Denoting the spatial interval of the source by $\Delta x_{production}$, we have the momentum uncertainty $\Delta p_{production} \propto 1/\Delta x_{production}$. and two mass eigenstates ν_k and ν_j are emitted coherently if $\left|p_k - p_j\right| \le \Delta p_{production}$. This condition imposes a condition for the squared mass difference of the neutrino states. If the momentum uncertainty in the production process is too small, different mass eignstates are emitted incoherently by the source and neutrinos do not oscillate. Implicitly, we have assumed that all the mass eigenstates are produced at the source at the same space-time point and detected at the same space-time point. Because both the production and detection processes have uncertainties in space and in time, different mass eigenstates can be produced and detected coherently at different location in space and time within the uncertainties. Thus, a similar discussion is necessary for the detection point. The maximal separation allowed for interference is:

$$\sigma_x^2 = \sigma_P^2 + \sigma_D^2 \equiv \left|\Delta x\right|^2_{\max} \approx \Delta x_P^2 + \Delta t_P^2 + \Delta x_D^2 + \Delta t_D^2$$

Comparing this transition probability with he results obtained in the frame of the plane wave formalism, we can see that two additional terms appear. The first factor

$$\exp\left[-\left(\frac{L}{L_{ab}^{coh}}\right)^2\right]$$

is connected with the fact that two wave packets, each with different momentum and energy, have slightly different group velocities. It means that after some time, or equivalently after a sufficient long distance between the source and the detector, the mass eigenstate wave packets do not longer overlap and cannot interfere to produce oscillations. Very important, a measurement can restore the coherence, because this limit distance is proportional with σ_x, (the precision of measurement of momenta in source and in detector).
The factor

$$\exp\left[-2\pi^2\rho^2\omega\left(\frac{\sigma_x}{L_{ab}^{osc}}\right)^2\right]$$

is approximately equal to unity if $\sigma_x << \left|L_{ab}^{osc}\right|$. This inequality must be satisfied to observe any oscillation. Also, on can see that to observe the oscillations, the localisation of the source and of the detector must be much better than the oscillation length. If $\sigma_x > \left|L_{ab}^{osc}\right|$ the wave packets of neutrinos a and b lose coherence, the oscillation between them is wash away.

3. The threefold maximal mixing hypothesis

3.1 Up-to-date constraints from experimental data

The literature is rife with both experimental and theoretical claims regarding neutrino properties, many of them in conflict with one another. The current experimental data suggest some conclusions, but these are not decisively established.

In the following discussion, we suppose only three neutrinos families and do not consider the existence of additional neutrinos, sterile otherwise.

From recent data of the Super-Kamiokande Collaboration [5] and from CHOOZ [6], atmospheric neutrinos rarely oscillate into electron neutrinos, the main argument being the L/E -flatness of the event ratio. If the presently observed results is confirmed, the possible interpretation is that the atmospheric muon neutrinos suffer maximal, or nearly maximal, two-flavour oscillations into tau neutrinos; and the relevant mixing angle satisfies the relation:

$$sin^2(2\vartheta) > 0.82$$

and the required neutrino mass-squared difference Δ_{23} satisfies the condition:

$$5\times10^{-4}eV^2 < \Delta_{23} < 6\times10^{-3}eV^2 .$$

Oscillations are needed to resolve the discrepancy between the observed and computed solar neutrino fluxes [7]. The first date from SNO experiment [8] have dramatically confirmed the long-standing HOMESTAKE solar neutrino experiment., SAGE, GALLEX and GNO experiments.

From this data, the relevant neutrino squared-mass difference is in the range:

$$6\times10^{-11}eV^2 < \Delta_{solar} < 2\times10^{-5}eV^2$$

The LSND experiment [9] claims direct evidence for $\nu_\mu \leftrightarrow \nu_e$ oscillations in a region which is partially unconstrained by other experiments.

If all the up-to-date constraints from solar, atmospheric, reactor, accelerator data are considered in the analysis of high energy neutrinos $\left(E \geq 10^6 GeV\right)$ which come from cosmologically distant astrophysical sources such as

Active Galactic Nuclei and Gamma Ray Burst fireball (typical distance is $100\ Mpc$), the allowed region [10] is a small area around the fluxes values: $\Phi(\nu_e) \cong \Phi(\nu_\mu) \cong \Phi(\nu_\tau) \cong \frac{1}{3}$.

Careful studies of many nuclear species have failed to detect neutrinoless double beta decay. These experiments provide bounds on M_{ee}, the ee component of the Majorana neutrino mass matrix in the charged lepton flavour basis - a weighted average of neutrino masses. The Heidelberg-Moskow experiment reported the Majorana nature of the electronic neutrino [11], but these results must be confirmed.

3.2 The threefold maximal mixing

Such mixing implies specific forms for the lepton mass matrices, corresponding to a cyclic permutation symmetry among the generations. In accord with Harrison, Perkins and Scott [12] the mixing matrix could be written as:

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega_1 & \omega_1 & \omega_1 \\ \omega_1 & \omega_2 & \omega_3 \\ \omega_1 & \omega_3 & \omega_2 \end{pmatrix}$$

where $UU^+ = 1$ and the ω_i $(i = 1,2,3)$ are the complex cube-root of unity.

In the maximal hypothesis, the mixing is maximal in that all the elements of the mixing matrix have equal modulus $|U_{ab}| = \frac{1}{\sqrt{3}}$.

In the plane wave formalism, the transition probabilities are:

$$|p|^2 = P(e \to e) = P(\mu \to \mu) = P(\tau \to \tau) = \\ = P(\bar{e} \to \bar{e}) = P(\bar{\mu} \to \bar{\mu}) = P(\bar{\tau} \to \bar{\tau}) = \frac{1}{3} + \frac{2}{9}(c_{12} + c_{23} + c_{31})$$

$$|q|^2 = P(e \to \mu) = P(\mu \to \tau) = P(\tau \to e) = \\ = P(\bar{e} \to \bar{\tau}) = P(\bar{\tau} \to \bar{\mu}) = P(\bar{\mu} \to \bar{e}) = \frac{1}{3} - \frac{1}{9}(c_{12} + c_{23} + c_{31}) + \frac{1}{\sqrt{3}}(s_{12} + s_{23} + s_{31})$$

$$|r|^2 = P(e \to \tau) = P(\tau \to \mu) = P(\mu \to e) = \\ = P(\bar{e} \to \bar{\mu}) = P(\bar{\mu} \to \bar{\tau}) = P(\bar{\tau} \to \bar{e}) = \frac{1}{3} - \frac{1}{9}(c_{12} + c_{23} + c_{31}) - \frac{1}{\sqrt{3}}(s_{12} + s_{23} + s_{31})$$

where: $c_{ki} = \cos\left(\frac{\Delta m_{ki}^2}{2} \cdot \frac{L}{E}\right)$ and $s_{ki} = sin\left(\frac{\Delta m_{ki}^2}{2} \cdot \frac{L}{E}\right)$.

In these hypotheses, the transition probabilities have the following properties:

For all L/E values, the requirement imposed by the conservation of the probability, $|p|^2 + |q|^2 + |r|^2 = 1$.

For any neutrino, the relation: $P(\nu \to \nu') = P(\bar{\nu}' \to \bar{\nu})$ is satisfied, consequence of the CPT invariance. In any disappearance experiment, the transition probability is a universal function of L/E. For appearance experiments, this implies the CP and T violation.

4. Results and discussions

In the wave packet formalism, in the threefold maximal mixing, the transition probabilities are:

$$|p_w|^2 = \frac{1}{3} + \frac{2}{9}\left(c_{12} f_{12} e_{12} + c_{23} f_{23} e_{23} + c_{31} f_{31} e_{31}\right)$$

$$|q_w|^2 = \frac{1}{3} - \frac{1}{9}\left(c_{12} f_{12} e_{12} + c_{23} f_{23} e_{23} + c_{31} f_{31} e_{31}\right) + \frac{1}{3\sqrt{3}}\left(c_{12} f_{12} e_{12} + c_{23} f_{23} e_{23} + c_{31} f_{31} e_{31}\right)$$

$$|r_w|^2 = \frac{1}{3} - \frac{1}{9}\left(c_{12} f_{12} e_{12} + c_{23} f_{23} e_{23} + c_{31} f_{31} e_{31}\right) - \frac{1}{3\sqrt{3}}\left(c_{12} f_{12} e_{12} + c_{23} f_{23} e_{23} + c_{31} f_{31} e_{31}\right)$$

where

$$f_{ki} = \exp\left[-2\pi^2 \rho^2 \omega \left(\frac{\sigma_x}{L_{ki}^{osc}}\right)^{"}\right] \quad \text{and} \quad e_{ki} = \exp\left[-\left(\frac{L}{L_{ki}^{coh}}\right)^2\right]$$

and the standard $"1 : \frac{1}{3} : \frac{5}{9}"$ probability oscillations prediction [12] is modified.

On can observe that the symmetry characteristics to plane wave formalism is lost, but the simple structure of these equations remains.

Detailed predictions depend on the character of the neutrino mass spectrum and the widths of the distributions. We also consider the case that the neutrino spectrum exhibits a pronounced hierarchy, similar to that of the charged leptons and the quarks, so that: $\Delta m_{23}^2 \cong -\Delta m_{31}^2 \equiv \Delta m^2$ and $\Delta m_{12}^2 \equiv \Delta m'^2 << \Delta m^2$ [1].

The analysis of the term $\left(\frac{\Delta m^{(')2}}{2}\right)^{-1}$ as a function of the ratio $\frac{L}{E}$ reduces the problems to the standard case analysed by Harrison and al. [12].

It is necessary to introduce some explicit considerations, to give any other prediction. We consider the production process, because in the majority of experiments neutrinos are produced in charged-current weak decays.

$$A^+ \to l_\alpha^+ + \nu_\alpha$$

where A^+ is the decaying particle and l_α^+ is the lepton state that determines the flavour of the produced neutrino. To be detected, charged-current or neutral-current weak processes are used, which have an energy threshold larger than some fraction of MeV. A possible process is:

$$\nu_\beta + X \to B + l_\beta^-,$$

generally with X at rest, at a space-time distance from the production process.

A possibility is the case when the production process is unlocalised, as for example the experiments with atmospheric neutrino and thus $\sigma_{\alpha,production} \to 0$. If the momentum of A^+ particle is determined with high accuracy, as in accelerator experiments, its position is practically unlocalised ($\sigma_{x,A^+} \to \infty$).

In most situations, the localisation of the A^+ and l_α^+ are the same order of magnitude and $\sigma_{x,A^+} \cong \sigma_{x,l_\alpha^+}$.

If the particles taking part to the detection process are unlocalised, the limit is $\sigma_{p,\det ection} \to 0$. If the X particle (at rest) is localised, but are unlocalised final detection particles, thus $\sigma_{pl_{\beta}^{-}} = 0$. In a realistic case, production and detection occur in matter and all the detection particles have uncertainties of the same order of magnitude, as well as all the production particles.

References

[1] B. Pontecorvo, Mesonium and Antimesonium, J. Exp.Teor. Phys. (USSR),33 (1957),549
B. Pontecorvo, Inverse Beta Processes and Nonconservation of Lepton Charge, J. Exp.Teor. Phys. (USSR),34 (1957),247
S. M. Bilenki, B. Pontecorvo, Phys. Rept. 41, (1978)225
Y. Takeuchi, Y. Tazaki, S. Y. Tsai, T. Yamazaki, hep-ph/9809558

[2] M. Zralek, Acta Phys. Polon. B29, (1998), 3925, hep-ph/9810543

[3] B. Kayser, Phys. Rev. D24 (1981)110

[4] C. Giunti, hep-ph/0205014; C. Giunti and C. W. Kim, hep-ph/0011074, C. Giunti, hep-ph/0105319

[5] T. Kajita, hep-ex/9807003

[6] M. Apollonio et. al., Phys. Lett. B420 (1998)397

[7] J. N. Bahcall, hep-ph/9807216

[8] Q. R. Ahmad et. al. Phys. Rev. Lett. 87 (2001) 87

[9] LSND Collaboration, see for example: C. Athanassopoulos et. al., Phys. Rev. C54 (1996)2685, Phys. Rev. Lett. 77 (1996)3082

[10] O. Yasuda, hep-ph/0005135

[11] H. V. Klapdor-Kleingrothaus et. al., Mod. Phy. Lett. A16 (2001)2409
H. V. Klapdor-Kleingrothaus et. al., Mod. Phys. Lett. A16 (2001)2469
H. V. Klapdor-Kleingrothaus et. al., Mod. Phys. Lett., in press, 2002

[12] P. F. Harrison, D. H. Perkins, W. G. Scott, Phys. Lett. B349(199)137