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THEORETICAL AND EXPERIMENTAL STUDY ON DETERMINATION OF THE ELASTIC PROPERTIES OF THE COMPOSITE MATERIALS

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Abstract. In this paper is presented a matriceal method for determination of elastic properties for composite materials. The method is based by imposing of some deformation and tension states, which verify the compatibility and continuation conditions on the separation surfaces between material's phases by experimental work it was established that in case of the longitudinal elastic modulus E, the results precision is higher than the results of elastic formula from the specially bibliography.

Key words: matriceal method, composite material, elastic modulus.

1. INTRODUCTION

Because in the evaluation of structures, in order totestimate the behavior of such constructions, computations must be made in foresight purposes, precise data are required on the mechanical behavior. All the alternatives we had in the investigation of the properties and of the behavior shown by composite materials knocked against the fact that a very large number of elastic constants have to be previously known, while the computations could be made only by the use of a large diversity of mechanical models [4]. The most representative theories starting from which the calculations of composite structures was attempted are [3]:

- The theory of the effective modules;
- The theory of the effective rigidities;
- The theory of meddlings;
- The theory of dislocations modified;
- The theory of the continuous microphases.

All these theories demonstrate that the mechanic and elastic properties of a composite material depend on many factors:

- The mechanical properties of their constituents;
- The volume ratio of their constituents;
- The geometrical setup of their constituents;
- The adhesion between the phases;
- The technological process employed for obtaining them.

This is why great computation difficulties show up when the mechanical behavior is studied and when the elastic coefficients are theoretically determined. The employed methods could be classified in three types:

- The one which find the extreme values, using variational theorems;
- The ones which find the exact solutions;
- The ones, which find semi-empiric approximations.

Haskin and Rosen [1] coped with determining the inferior, respectively superior limits of the elasticity modules. They studied reinforced composites with long fibres, of a constant diameter, hexagonally arranged. Aboudi [2] studied following the same direction, but using a square elementary cell, where the section of the reinforcing piece is square-shaped. Cederbaum [5] achieved a micro-mechanical analysis of the composite material, which allowed the determining of its behavior as a whole, starting from the known properties of its individual constituents and from the details of their interaction. The method he used presents the advantages that there are no apriori hypotheses, these being generated during the analysis itself. The result is a macro-mechanical evaluation, as the homogeneous medium seen as transversally isotropic, thereby its properties being evaluable for all the possible combinations between fibres and matrices. In [9] are presented particular cases.

In order to determine the elastic coefficients for composites, the method of the finite element was applied too. We could mention the works of Huang and Hu [7], where is analyzed the case of the properties of materials bearing spherical, inclusions, Meguid [6], where are determined the properties of a composite bar with long fibres that are arranged twice periodically in the section, respectively Pyrez [8], who performs a probabilistic analysis of these properties.

2. THEORETICAL CONSIDERATIONS

Let us consider a sandwich-shaped composite formed of strata of constant thickness (fig.1). A referential system $O_{X_1X_2X_3}$ is then chosen so that the plane $O_{X_1X_2}$ should be parallel to the separation planes between the strata. For each strata, let us consider a tension state:

$$\left\{\sigma^{(i)}\right\}=\left\{\sigma_{11}^{i};\sigma_{22}^{i};\sigma_{12}^{i};\sigma_{23}^{i};\sigma_{13}^{i};\sigma_{33}^{i}\right\}^{t},\ i=\overline{1,n} \qquad (1)$$

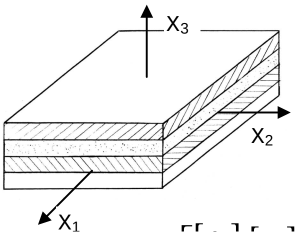

Fig.1 - Sandwich - shaped composite

The components of the tension state have to satisfy to the conditions. Through the component equations, the deformation states for the constituent materials are determined:

$$\begin{bmatrix} & \\ [B_i]^{t} & [C_i] \end{bmatrix}\Big\},\ i=\overline{1,n} \qquad (2)$$

in which:

- $\left\{\sigma^{(i)}\right\}=\left\{\sigma_{11}^{i};\sigma_{22}^{i};\sigma_{12}^{i};\gamma_{23}^{i};\gamma_{13}^{i};\gamma_{33}^{i}\right\}^{t},\ i=\overline{1,n}$ (3)

is the deformation state for the material "i" and $[A_i]$, $[B_i]$, $[C_i]$ are cells of the supplements matrix.

The deformation states must satisfy to the compatibility conditions of Saint–Venant. On the separation surfaces between the strata, continuity conditions for deformations and tensions must be satisfied too. So, on the separation surface between stratum "i" and stratum "i+1" the following continuity conditions must be verified:

- For tensions

$$\begin{aligned} &\sigma_{33}^{(i)}=\sigma_{33}^{(i+1)}=\sigma_{33} \\ &\sigma_{23}^{(i)}=\sigma_{23}^{(i+1)}=\sigma_{23} \\ &\sigma_{13}^{(i)}=\sigma_{13}^{(i+1)}=\sigma_{13} \qquad i=1 \end{aligned} \qquad (4)$$

- For deformations:

$$\begin{aligned} &\varepsilon_{11}^{(i)}=\varepsilon_{11}^{(i+1)}=\varepsilon_{11} \\ &\varepsilon_{22}^{(i)}=\varepsilon_{22}^{(i+1)}=\varepsilon_{22} \qquad i=\overline{1,n-1} \\ &\gamma_{12}^{(i)}=\gamma_{12}^{(i+1)}=\gamma_{12} \end{aligned} \qquad (5)$$

Through these, the states of tensions and deformations, given by (1) and (3), could be written under the form:

$$\left\{\sigma^{(i)}\right\}=\begin{Bmatrix}\{\sigma_i\}\\ \{\sigma\}\end{Bmatrix} \qquad i=\overline{1,n} \tag{6}$$

$$\left\{\varepsilon^{(i)}\right\}=\begin{Bmatrix}\{\varepsilon\}\\ \{\varepsilon_i\}\end{Bmatrix} \qquad i=\overline{1,n} \tag{7}$$

Where:

$$\left\{\sigma_i\right\}=\left\{\sigma_{11}^{(i)};\sigma_{22}^{(i)};\sigma_{12}^{(i)}\right\}^t \quad i=\overline{1,n} \tag{8}$$

$$\left\{\sigma\right\}=\left\{\sigma_{23};\sigma_{13};\sigma_{33}\right\}^t$$

Respectively:

$$\left\{\varepsilon_i\right\}=\left\{\gamma_{23}^{(i)};\gamma_{13}^{(i)};\varepsilon_{33}^{(i)}\right\}^t \qquad i=\overline{1,n} \tag{9}$$

$$\left\{\varepsilon\right\}=\left\{\varepsilon_{11};\varepsilon_{22};\gamma_{12}\right\}^t$$

From the constituent equations (2), we obtain:

$$\left\{\sigma_i\right\}=\left[A_i\right]^{-1}\left\{\varepsilon\right\}-\left[A_i\right]^{-1}\left[B_i\right]\left\{\sigma\right\},\ i=\overline{1,n} \tag{10}$$

$$\left\{\varepsilon_i\right\}=\left[B_i\right]^t\left[A_i\right]^{-1}\left\{\varepsilon\right\}+\left(\left[C_i\right]-\left[B_i\right]^t\left[A_i\right]^{-1}\left[B_i\right]\right)\left\{\sigma\right\},\ i=\overline{1,n} \tag{11}$$

The matrix of the average deformations $\left\{\overline{\varepsilon}\right\}$ and the matrix of average tensions $\left\{\overline{\sigma}\right\}$ are defined by the relations:

$$\left\{\overline{\varepsilon}\right\}=\frac{1}{v}\sum_{i=1}^{n}\iiint_{(v_i)}\left\{\varepsilon^{(i)}\right\}d\tau\,, \tag{12}$$

$$\left\{\overline{\sigma}\right\}=\frac{1}{v}\sum_{i=1}^{n}\iiint_{(v_i)}\left\{\sigma^{(i)}\right\}d\tau\,, \tag{13}$$

where V is the total volume, and V_i is the volume of stratum “i”.

We can consider in particular that the matrices $\left\{\varepsilon\right\}$and $\left\{\sigma\right\}$ have constant components (which means that the Cauchy conditions for deformations are satisfied, as well as the Saint-Venant conditions for deformations). In this case, from (12) and (13), for the matrices of average tensions, respectively average deformations, we obtain the expression:

$$\left\{\overline{\sigma}\right\}=\begin{bmatrix}[\alpha] & -[\beta]\\ [0] & [I]\end{bmatrix}\begin{Bmatrix}\{\varepsilon\}\\ \{\sigma\}\end{Bmatrix} \tag{14}$$

Respectively:

$$\{\bar{\varepsilon}\}=\begin{bmatrix}[I] & [0]\\ [\beta]^{t} & [\gamma]\end{bmatrix}\begin{Bmatrix}\{\varepsilon\}\\ \{\sigma\}\end{Bmatrix} \tag{15}$$

Where:

- [0] is the naught matrixes;
- [l] is the unity matrix;
- $[\alpha]=\sum_{i=1}^{n}V_i[A_i]^{-1}$ (16)
- $[\beta]=\sum_{i=1}^{n}V_i[A_i]^{-1}$ (17)
- $[\gamma]=\sum_{i=1}^{n}V_i\left([C_i]-[B_i]^{t}[A_i]^{-1}[B_i]\right)$ (18)

Into which:

$$V_i=\frac{v_i}{v}\quad i=\overline{1,n} \tag{19}$$

is the volume ratio of the stratum "i".

The constituent equation of the composite supposes a relation between the matrix of the average tensions and the matrix of the average deformation. This could be written under the following matrix form: $\{\bar{\varepsilon}\}=\begin{bmatrix}[A] & [B]\\ [B'] & [C]\end{bmatrix}\{\bar{\sigma}\}$

(20)

By the substitutions, offered through (14) and (15), of the matrices of the average tensions and deformations into (20), then by appropiate identifications, the expression for the cells are obtained $[A],[B],[B'],[C]$:

$$\begin{aligned}&[A]=[\alpha]^{-1}\\&[B]=[\alpha]^{-1}[\beta]\\&[B']=[\beta]^{t}[\alpha]^{-1}\\&[C]=[\gamma]+[\beta]^{t}[\alpha]^{-1}[\beta]\end{aligned} \tag{21}$$

From the analysis of the relations (21), it results that the composite's elastic properties are depending on the volume ratios, on the distribution in space and on the constituents' properties. While the relations previously found in the specialty's literature treat separately every elastic constant, the relation (21) is totally determining the matrix of the elastic constants, by tacking into account the pre-existing connections between them. Because the matrices $[\alpha]$ and $[\gamma]$ are symmetrical and $[B']=[B]^{t}$, it results that , for the composite material, the matrix of the elastic coefficients is symmetrical too.

3. PARTICULAR CASES

If the materials from the composites strata should be orthotropic, then the cells of the matrices for elastic constants would have the forms:

$$[A_i]=\begin{bmatrix} S_{11}^{(i)} & S_{12}^{(i)} & 0 \\ S_{21}^{(i)} & S_{22}^{(i)} & 0 \\ 0 & 0 & S_{66}^{(i)} \end{bmatrix},\quad [B_i]=\begin{bmatrix} 0 & 0 & S_{13}^{(i)} \\ 0 & 0 & S_{23}^{(i)} \\ 0 & 0 & 0 \end{bmatrix}$$

$$[C_i]=\begin{bmatrix} S_{44}^{(i)} & 0 & 0 \\ 0 & S_{55}^{(i)} & 0 \\ 0 & 0 & S_{33}^{(i)} \end{bmatrix} \qquad i=\overline{1,n} \tag{22}$$

In this case, through (21) we come to:

$$[A]=\begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix},\quad [B]=\begin{bmatrix} 0 & 0 & S_{13} \\ 0 & 0 & S_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

$$[C]=\begin{bmatrix} S_{44} & 0 & 0 \\ 0 & S_{55} & 0 \\ 0 & 0 & S_{33} \end{bmatrix} \tag{23}$$

Into which:

$$S_{11}=\frac{1}{\Delta}\sum_{i=1}^{n}\frac{V_i}{\Delta_i}S_{11}^{(i)} \tag{24}$$

$$S_{12}=\frac{1}{\Delta}\sum_{i=1}^{n}\frac{V_i}{\Delta_i}S_{12}^{(i)} \tag{25}$$

$$S_{21}=\frac{1}{\Delta}\sum_{i=1}^{n}\frac{V_i}{\Delta_i}S_{21}^{(i)} \tag{26}$$

$$S_{22}=\frac{1}{\Delta}\sum_{i=1}^{n}\frac{V_i}{\Delta_i}S_{22}^{(i)} \tag{27}$$

$$S_{13}=\frac{1}{\Delta}\left(\sum_{i=1}^{n}V_i\frac{S_{11}^{(i)}}{\Delta_i}\right)\left[\sum_{i=1}^{n}\frac{V_i}{\Delta_i}\left(S_{22}^{(i)}S_{13}^{(i)}-S_{12}^{(i)}S_{23}^{(i)}\right)\right]+\frac{1}{\Delta}\left(\sum_{i=1}^{n}\frac{V_i}{\Delta_i}S_{12}^{i}\right) \left[\sum_{i=1}^{n}\frac{V_i}{\Delta_i}\left(S_{11}^{(i)}S_{23}^{(i)}-S_{21}^{(i)}S_{13}^{(i)}\right)\right] \tag{28}$$

$$S_{23} = \frac{1}{\Delta}\left(\sum_{i=1}^{n}\frac{V_i}{\Delta_i}S_{21}^{(i)}\right)\left[\sum_{i=1}^{n}\frac{V_i}{\Delta_i}\left(S_{22}^{(i)}S_{13}^{(i)} - S_{12}^{(i)}S_{23}^{(i)}\right)\right] + \frac{1}{\Delta}\left(\sum_{i=1}^{n}\frac{V_i}{\Delta_i}S_{22}^{i}\right) \left[\sum_{i=1}^{n}\frac{V_i}{\Delta_i}\left(S_{11}^{(i)}S_{23}^{(i)} - S_{21}^{(i)}S_{13}^{(i)}\right)\right] \tag{29}$$

$$S_{66} = \left[\sum_{i=1}^{n}\frac{V_i}{S_{66}^{(i)}}\right]^{-1} \tag{30}$$

$$S_{44}^{(i)} = \sum_{i=1}^{n}V_i S_{44}^{(i)} \tag{31}$$

$$S_{55}^{(i)} = \sum_{i=1}^{n}V_i S_{55}^{(i)} \tag{32}$$

$$S_{33}^{(i)} = \sum_{i=1}^{n}V_i\left(S_{33}^{(i)} - \alpha_i\right) + S_{13}\left[\sum_{i=1}^{n}\frac{V_i}{\Delta_i}\left(S_{22}^{(i)}S_{13}^{(i)} - S_{12}^{(i)}S_{23}^{(i)}\right)\right] + S_{23}\left[\sum_{i=1}^{n}\frac{V_i}{\Delta_i}\left(S_{23}^{(i)}S_{11}^{(i)} - S_{13}^{(i)}S_{21}^{(i)}\right)\right] \tag{33}$$

Where:

$$\Delta_i = S_{11}^{(i)}S_{22}^{(i)} - S_{12}^{(i)}S_{21}^{(i)} \tag{34}$$

$$\Delta = \left(\sum_{i=1}^{n}\frac{V_i}{\Delta_i}S_{22}^{(i)}\right)\left(\sum_{i=1}^{n}\frac{V_i}{\Delta_i}S_{11}^{(i)}\right) - \left(\sum_{i=1}^{n}\frac{V_i}{\Delta_i}S_{12}^{(i)}\right)\left(\sum_{i=1}^{n}\frac{V_i}{\Delta_i}S_{21}^{(i)}\right) \tag{35}$$

$$\alpha_i = \frac{1}{\Delta_i}\left[S_{13}^{(i)}\left(S_{22}^{(i)}S_{13}^{(i)} - S_{12}^{(i)}S_{23}^{(i)}\right) + S_{23}^{(i)}\left(S_{11}^{(i)}S_{23}^{(i)} - S_{21}^{(i)}S_{13}^{(i)}\right)\right] \tag{36}$$

If the composite's strato should have been made of homogeneous and isotropic materials, then the cells of the matrices representing the elastic coefficients would have had the form:

$$[A_i] = \begin{bmatrix} \frac{1}{E_i} & -\frac{\nu_i}{E_i} & 0 \\ -\frac{\nu_i}{E_i} & \frac{1}{E_i} & 0 \\ 0 & 0 & \frac{1}{E_i} \end{bmatrix} \qquad [B_i] = \begin{bmatrix} 0 & 0 & -\frac{\nu_i}{E_i} \\ 0 & 0 & -\frac{\nu_i}{E_i} \\ 0 & 0 & 0 \end{bmatrix}$$

$$[C_i]=\begin{bmatrix} \frac{1}{G_i} & 0 & 0 \\ 0 & \frac{1}{G_i} & 0 \\ 0 & 0 & \frac{1}{E_i} \end{bmatrix} \qquad i=\overline{1,n} \tag{37}$$

Where:

- E_i is Young's modulus;
- G_i is the tearing modulus for the material ″i″;
- ν_i Poisson's coefficient for the material ″i″.

The cells of the matrix for elastic coefficients have the form (23), where:

$$S_{11}=S_{22}=\frac{1}{\Delta}\sum_{i=1}^{n}\frac{V_iE_i}{1-\nu_i^2} \tag{38}$$

$$S_{21}=S_{12}=-\frac{1}{\Delta}\sum_{i=1}^{n}\frac{V_iE_i\nu_i}{1-\nu_i^2} \tag{39}$$

$$S_{13}=S_{23}=-\frac{1}{\Delta}\left(\sum_{i=1}^{n}\frac{V_i\nu_i}{1-\nu_i}\right)\left(\sum_{i=1}^{n}\frac{V_iE_i}{1+\nu_i}\right) \tag{40}$$

$$S_{66}=\frac{1}{\sum_{i=1}^{1}V_iG_i} \tag{41}$$

$$S_{44}=S_{55}=\sum_{i=1}^{n}\frac{V_i}{G_i} \tag{42}$$

$$S_{33}=\sum_{i=1}^{n}\frac{V_i(1+\nu_i)(1-2\nu_i)}{E_i(1-\nu_i)}+\frac{2}{\Delta}\left(\sum_{i=1}^{n}\frac{V_i\nu_i}{1-\nu_i}\right)^2\left(\sum_{i=1}^{n}\frac{V_iE_i}{1+\nu_i}\right) \tag{43}$$

în care:

$$\Delta=\left(\sum_{i=1}^{n}\frac{V_i\nu_i}{1-\nu_i}\right)^2-\left(\sum_{i=1}^{n}\frac{V_i\nu_iE_i}{1-\nu_i{}^2}\right)^2 \tag{44}$$

4. EXPERIMENTAL RESEARCHES

The most important elastic constant of a material is the elasticity modulus. In the view of its determination, a device was used, presenting the loading scheme from figure 2, on which the test boards were fixed.

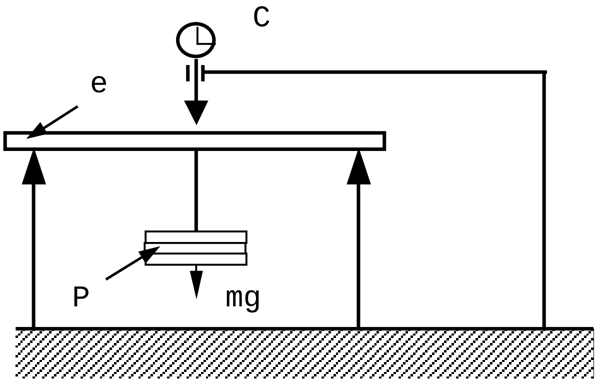

Fig.2. Experimental setup: (e)- test board; (C)-comparative apparatus (P)-external loading

On them were exerted successive charges, and the arrow was measured with the help of a comparative apparatus.

The form and the dimensions of the best boards are presented in figure 3.

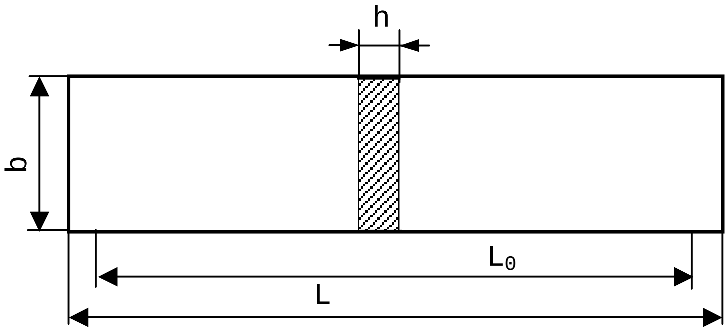

Fig.3. - The form and the dimensions of the best boards.

For the calculation of the elasticity modulus was employed the formula:

$$E = \frac{L_0^3}{4bh^3} \cdot \frac{\Delta P}{\Delta f} \; [\mathrm{MPa}] \qquad (45)$$

Where:

- L_o = the length existing between the supports [mm];
- b = the width of the test board [mm];
- h = the thickness of the test board [mm];
- ΔP = the variation of the external loading [N];
- Δf = the variation of the arrow [mm].

Measurements were made for test boards made of resin (E=4500MPa;$\nu = 0{,}3$) reinforced with fibbers of flax (E=28000 MPa, $\nu = 0{,}2$), for three volume ratios of the reinforcer, the results being presented in table (1).

Table 1. Experimental modulus of elasticity

Nr crt	Volume ratio		Dimensions of test board			Experimental data		Modulus of elasticity E [Mpa]
	Flax	Resin	L_0 [mm]	b [mm]	h [mm]	ΔP [N]	Δf [mm]	
1	0.4	0.6	205	26.2	2.8	0.98	0.29	12917
						1.96	0.57	
						2.94	0.87	
2	0.5	0.5	205	25.5	2.5	0.98	0.36	15451
						1.96	0.68	
						2.94	1.06	
3	0.6	0.4	205	26.5	2.6	0.98	0.26	17709
						1.96	0.52	
						2.94	0.79	

The most used formula in the calculation of the elasticity modulus at the bi-phase composite materials is [9;10.]:

$$E_L = E_1 V_1 + E_2 V_2 \tag{46}$$

From the relation (32), the elasticity modulus could be obtained for the composite material made of two constituents:

$$E = \frac{1}{S_{11}} = \frac{\left(V_1 E_1 + V_2 E_2\right)^2 - \left(V_1 E_1 \nu_2 + V_2 E_2 \nu_1\right)^2}{V_1 E_1\left(1-\nu_2^2\right) + V_2 E_2\left(1-\nu_1^2\right)} \tag{47}$$

In the table (2) are comparatively presented the results obtained through the relations (46) and (47) and the experimental ones.

Table 2. Comparative results of the modulus of elasticity

No. Crt.	Volume ratio		Modulus of elasticity E [MPa]		
	V_1	V_2	Relation (46)	Relation (47)	Experimental results
1	0,4	0,6	13900	13480	12917
2	0,5	0,5	16250	15640	15451
3	0,6	0,4	18600	17830	17709

5. CONCLUSIONS

In the proposed method, the unhomogeneous material, formed by homogeneous phases, is assimilated with a homogeneous, but unisotropic material.

Unlike the relations from the specialty reference material, which discuses separately each elastic constant, in relations (21) it determines unitary all elastic constants.

It also proves that elastic coefficients matrix of the composite material is symmetrically.

Using the calculations relations the matrix elements of the composite's elastic constituents, it can be observed that they depend on:

- the volumic proportion of the constituents;

- the volumic distribution (the spacial distribution) of the constituents;
- elastic properties of the component phases.

Since the method principle consist in weighting of the deformation states and of the strains, and of determination the relation between them, we can conclude that composite's properties also depend on solicitation manner to which is submission, particularly, on the elaboration method.

By the comparison of the theoretical and the experimental results, it results that for the longitudinal elasticity modulus, the theoretical relations, both deduced and from reference material overestimates its numeric value.

This can be explained that it cannot entirely eliminate the air pockets. This makes that, in real, the volumic proportions of the constituents being smaller than the theoretical ones.

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