

HOMOGENIZATION RESULTS FOR PARABOLIC PROBLEMS WITH DYNAMICAL BOUNDARY CONDITIONS

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Abstract. The aim of this paper is to study the asymptotic behavior of the solution of a parabolic dynamical boundary-value problem in a periodically perforated domain. The domain is considered to be a fixed bounded open subset $\Omega \subset \mathbf{R}^N$, in which identical and periodically distributed perforations (holes) of size ε are made. In the perforated domain we consider a heat equation, with a Dirichlet condition on the exterior boundary and a dynamical boundary condition on the surface of the holes. The limit equation, as $\varepsilon \rightarrow 0$, is a heat equation with constant coefficients, but with extra-terms coming from the influence of the non-homogeneous dynamical boundary condition.

Key words: homogenization, energy method, dynamical boundary-value problems.

INTRODUCTION

The aim of this paper is to study the asymptotic behavior of the solution of a parabolic dynamical boundary-value problem in a periodically perforated domain. Such problems, although not too widely considered in the literature, are very natural in many mathematical models as partially saturated flows in porous media, heat transfer in a solid in contact with a moving fluid, diffusion phenomena in porous media (see [1], [2] and the references therein).

Let Ω be an open bounded set in \mathbf{R}^N and let us perforate it by holes. As a result, we obtain an open set Ω^ε which will be referred to as being the *perforated domain*; ε represents a small parameter related to the characteristic size of the perforations. We shall deal with the case in which the perforations (holes) are identical and periodically distributed and their size is of the order of ε . In the perforated domain we consider a heat equation, with a Dirichlet condition on the exterior boundary and a dynamical boundary condition on the surface of the holes.

Our main motivation is to study the asymptotic behavior, as $\varepsilon \rightarrow 0$, of the solution u^ε of the following dynamical boundary-value problem:

$$\frac{\partial u^e}{\partial t} - \Delta u^e = f(t, x), \quad \text{in } \Omega^e \times (0, T), \quad (1)$$

$$\frac{\partial u^e}{\partial n} + \mathbf{e} \frac{\partial u^e}{\partial t} = \mathbf{e}g(t, x), \quad \text{on } \partial F^e \times (0, T), \quad (2)$$

$$u^e(0, x) = u^0(x), \quad \text{in } \Omega^e, \quad (3)$$

$$u^e(0, x) = v^0(x), \quad \text{on } \partial F^e, \quad (4)$$

$$u^e = 0 \quad \text{on } \partial\Omega \times (0, T). \quad (5)$$

Here, $f \in L^2(0, T, L^2(\Omega))$, $g \in L^2(0, T, H_0^1(\Omega))$, $u^0 \in L^2(\Omega)$, $v^0 \in L^2(\partial F^e)$, $(0, T)$ is the time interval of interest and ∂F^e is the boundary of the holes.

As we shall see in Section 4, there exists an extension \tilde{u}^e of u^e into all $\Omega \times (0, T)$ such that $\tilde{u}^e \rightarrow u$ strongly in $L^2(0, T, L^2(\Omega))$ and u is the unique solution of the following system (the macromodel):

$$\left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \frac{\partial u}{\partial t} - \quad (6)$$

$$\nabla(Q\nabla u) = \frac{|Y^*|}{|Y|} f + \frac{|\partial F|}{|Y|} g, \quad \text{in } \Omega \times (0, T), \quad (7)$$

$$u(0, x) = u^0(x), \quad \text{in } \Omega, \quad (8)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T).$$

where $Q = ((q_{ij}))$ is the classical homogenized matrix whose entries are defined by

$$q_{ij} = \frac{|Y^*|}{|Y|} \mathbf{d}_{ij} - \frac{1}{|Y|} \int_{Y^*} \frac{\partial \mathbf{h}_j}{\partial y_i} dy$$

in terms of the functions \mathbf{h}_j , solutions of the system

$$-\Delta \mathbf{h}_j = 0, \quad \text{in } Y^*, \quad (9)$$

$$\frac{\partial(\mathbf{h}_j - y_j)}{\partial n} = 0, \quad \text{on } \partial F, \quad (10)$$

$$\mathbf{h}_j \text{ is } Y\text{-periodic.} \quad (11)$$

Here, F is the elementary hole, Y is the elementary cell of periodicity and $Y^* = Y \setminus \overline{F}$ (see Section 2). Thus, in the limit, when $\epsilon \rightarrow 0$, we get a classical constant coefficient heat equation, with a Dirichlet boundary condition, with a non-homogeneous right-hand term and with a constant (due to the periodicity) extra-term in front of the time derivative, coming from the well-balanced contribution of the dynamical part of our boundary condition on the surface of the holes. Also, let us note that the external force g acting on the boundary of the holes leads in the limit to a force distributed all over the domain $\tilde{\Omega}$.

The plan of the paper is the following one: in the second section we introduce some useful notations and assumptions. In Section 3 we give the main convergence result of this paper, i.e. the macromodel. For obtaining this macromodel, we need some preliminary results, which are given in Section 4. The last section is devoted to the proof of some a priori estimates, independent of ϵ for the solution of the micromodel and to the proof of the convergence result.

Problems closed to this one have been considered by many authors. Among others, let us mention the papers of D. Cioranescu and P. Donato [4], [5], C. Conca and P. Donato [9]. The homogenization of Laplace and Poisson equations in perforated domains with holes of the same size as the period and with homogeneous Dirichlet conditions on the surface of the holes and on the exterior boundary of the domain was treated in [7]. The same problem, but with homogeneous Neumann boundary conditions on the holes, was treated in [5]. For the non-homogeneous case, we can refer to [4]. The homogenization of the Poisson equation (or even a more general elliptic equation) with non-homogeneous Fourier boundary conditions on surface of the holes has been treated in [5].

NOTATION AND ASSUMPTIONS

Let $\tilde{\Omega}$ be a bounded connected open set in \mathbf{R}^N , with boundary $\partial\tilde{\Omega}$ of class C^2 and let $[0, T]$ be the time interval of interest.

Let $Y = [0, l_1[\times]0, l_2[\times \dots \times]0, l_N[$ be the representative cell in \mathbf{R}^N and F an open subset of Y with boundary ∂F of class C^2 , such that $\overline{F} \subset Y$. We shall denote by $F^{\epsilon, k}$ the translated image of ϵF by ϵk , $k \in \mathbf{Z}^N$. Also, we shall denote by F^ϵ the set of all the holes contained in Ω and by $\Omega^\epsilon = \Omega \setminus \overline{F^\epsilon}$. Hence, Ω^ϵ is a periodically perforated domain with holes of the same size as the period. Let us remark that the holes do not intersect the boundary $\partial\Omega$.

We shall also use the following notations:

$$Y^* = Y \setminus \overline{F}, \quad \mathbf{q} = \frac{|Y^*|}{|Y|}.$$

Also, we shall denote by \mathbf{c}^ϵ the characteristic function of the domain Ω^ϵ .

Let us introduce the following usual function spaces and norms:

$$H = L^2(\Omega), (u, v)_\Omega = \int_\Omega uv dx,$$

$$\bar{H} = L^2(0, T; H), (u, v)_{\Omega, T} = \int_0^T (u(t), v(t))_\Omega dt,$$

$$V = H^1(\Omega), (u, v)_V = (u, v)_\Omega + (\nabla u, \nabla v)_\Omega,$$

$$\bar{V} = L^2(0, T; V), (u, v)_V = \int_0^T (u(t), v(t))_V dt.$$

THE CONVERGENCE RESULT

For $f \in L^2(0, T; L^2(\Omega))$, $g \in L^2(0, T; H_0^1(\Omega))$, $u^0 \in L^2(\Omega)$, $v^0 \in L^2(\partial F^e)$, there exists a unique solution u^e of the problem (1)-(5) (see, for instance, [3] and [11]-[12]).

The main result of this paper is the following one:

Theorem 3.1. Let u^e be the unique solution of the problem (1)-(5). Then, there exists an extension \tilde{u}^e of u^e into all $\Omega \times (0, T)$ such that $\tilde{u}^e \rightarrow u$ strongly in \bar{H} and u is the unique solution of the following system (the macromodel):

$$\left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \frac{\partial u}{\partial t} - \nabla(Q \nabla u) = \frac{|Y^*|}{|Y|} f + \frac{|\partial F|}{|Y|} g, \quad \text{in } \Omega \times (0, T), \quad (12)$$

$$u(0, x) = u^0(x), \quad \text{in } \Omega, \quad (13)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T). \quad (14)$$

where $Q = ((q_{ij}))$ is the classical homogenized matrix whose entries are defined by

$$q_{ij} = \frac{|Y^*|}{|Y|} \mathbf{d}_{ij} - \frac{1}{|Y|} \int_{Y^*} \frac{\partial \mathbf{h}_j}{\partial y_i} dy \quad (15)$$

in terms of the functions \mathbf{h}_j , solutions of the system

$$-\Delta \mathbf{h}_j = 0, \quad \text{in } Y^*, \quad (16)$$

$$\frac{\partial(\mathbf{h}_j - y_j)}{\partial n} = 0, \quad \text{on } \partial F, \quad (17)$$

$$\mathbf{h}_j \text{ is } Y\text{-periodic.} \quad (18)$$

Thus, in the limit, when $\mathbf{e} \rightarrow 0$, we get a classical constant coefficient heat equation, with a Dirichlet boundary condition, with a non-homogeneous right-hand term and with a constant (due to the periodicity) extra-term in front of the time derivative, coming from the well-balanced contribution of the dynamical part of our boundary condition on the surface of the holes

Let us note that, using a standard regularization procedure, we are allowed to suppose that we have a better regularity of the data. Hence, we can get a more regular solution ([3], [13]). More precisely, we can suppose that

$$u^0 \in H^2(\Omega) \cap H_0^1(\Omega), \quad v^0 = u^0|_{\partial\Omega^e}, \quad f \in C^1([0, T], L^2(\Omega)), \\ g \in C^1([0, T], H_0^1(\Omega)).$$

Then, our solution satisfies:

$$u^e \in C([0, T]; H^2(\Omega^e)) \cap H_{\partial\Omega}^1(\Omega^e) \cap C^1([0, T]; L^2(\Omega^e)), \\ u_t^e \in C((0, T); H_{\partial\Omega}^1(\Omega^e)).$$

For such regular solutions, the weak formulation of the system (1)-(5) is the following one:

$$\text{Find } u^e \in C([0, T]; H^2(\Omega^e)) \cap H_{\partial\Omega}^1(\Omega^e) \cap C^1([0, T]; L^2(\Omega^e)),$$

$$u^e(0) = u^0|_{\Omega^e} \text{ such that}$$

$$-\left(u^e, \frac{\partial \mathbf{j}}{\partial t}\right)_{\Omega^e, T} + (\nabla u^e, \nabla \mathbf{j})\Big|_{\Omega^e, T} - \mathbf{e}\left(u^e, \frac{\partial \mathbf{j}}{\partial t}\right)_{\partial F^e, T} = \\ = (f, \mathbf{j})_{\Omega^e, T} + \mathbf{e}(g, \mathbf{j})_{\partial F^e, T}, \quad (19)$$

$$\text{for any } \mathbf{j} \in \overline{D} = C_0^\infty((0, T) \times \Omega).$$

PRELIMINARY RESULTS

For obtaining the macromodel, we have to pass to the limit, with $\mathbf{e} \rightarrow 0$, in some surface integrals on the boundary of the holes. For doing this, we shall make use of a convergence result based on a technique introduced by M. Vanninathan [13], which transforms surface integrals into volume integrals. This method was also used for the elliptic case in [4] and [6]. For a given function $h \in L^2(\partial F)$, following [4], let us denote

$$C_h = \frac{|\partial F|}{|Y^*|} M_{\partial F}(h), \quad (20)$$

where $M_{\partial F}(h)$ is the mean value of h over ∂F . Also, let

$$\mathbf{m}_h = qC_h. \quad (21)$$

In particular, $C_1 = \frac{|\partial F|}{|Y^*|}$, $\mathbf{m}_1 = \frac{|\partial F|}{|Y|}$.

For $h \in L^2(\partial F)$, we define the measure \mathbf{m}_h^e by

$$\langle \mathbf{m}_h^e, \mathbf{j} \rangle = \mathbf{e} \int_{\partial F^e} h\left(\frac{x}{\mathbf{e}}\right) \mathbf{j}(x) d\mathbf{S}, \quad \text{for any } \mathbf{j} \in H_0^1(\Omega). \quad (22)$$

In [4] it was proved that $\mathbf{m}_h^e \rightarrow \mathbf{m}_h$ strongly in $H^{-1}(\Omega)$, with \mathbf{m}_h given by (24). Moreover, if h is constant and the boundary of F is smooth, the above convergence takes place strongly in $W^{-1,\infty}(\Omega)$. In the general case, when $h \in L^2(\partial F)$, it was proved in [6] that if $w^{\hat{a}}$ is a sequence from $H_0^1(\Omega)$ which converges weakly in $H_0^1(\Omega)$ to w , then the corresponding linear form \mathbf{m}_h^e defined by (25) on $H^1(\Omega^e)$ satisfies:

$$\langle \mathbf{m}_h^e, w^e \big|_{\Omega^e} \rangle \rightarrow \mathbf{m}_h \int_{\Omega} w dx. \quad (23)$$

On the other hand, since the solution of the problem (1)-(5) is defined only in Ω^e , we need to extend it to the whole Ω . For finding a suitable extension \tilde{u}^e into all Ω , we shall use the following well-known extension lemma (see, for instance, [7]):

Lemma 4.1. (i) Any function $\mathbf{j} \in H^1(Y^*)$ can be extended to a function $\tilde{\mathbf{j}} \in H^1(Y)$ such that

$$\|\nabla \tilde{\mathbf{j}}\|_Y \leq C \|\nabla \mathbf{j}\|_{Y^*}.$$

(ii) Any function $\mathbf{j}^e \in H^1(\Omega^*)$ can be extended to a function $\tilde{\mathbf{j}}^e \in H_0^1(\Omega)$ such that

$$\|\nabla \tilde{\mathbf{j}}^e\|_{\Omega} \leq C \|\nabla \mathbf{j}^e\|_{\Omega^*},$$

where C is a constant independent of \mathbf{e} .

PROOF OF THE CONVERGENCE RESULT

In this section we shall prove the convergence result given by Theorem 3.1. for the solution of the problem (1)-(5). This solution being defined only on Ω^e , we need

to extend it to the whole of Ω in order to be able to state a convergence result. For doing this, we shall need some accurate estimates for the solution u^ϵ , independent of ϵ .

Proof of Theorem 3.1. It is enough to prove the convergence result given by Theorem 3.1. for regular data and so, for regular solutions u^ϵ . Then, by performing a classical regularization process of our data, we are able to prove immediately the convergence result for the general case. Hence, let $u^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $v^0 = u^0|_{\partial\Omega^\epsilon}$.

Also, let $f \in C^1([0, T], L^2(\Omega))$ and $g \in C^1([0, T], H_0^1(\Omega))$. For such regular data we know that there exists a unique solution of the problem (1)-(5) such that

$$u^\epsilon \in C([0, T]; H^2(\Omega^\epsilon)) \cap H_{\partial\Omega}^1(\Omega^\epsilon) \cap C^1([0, T]; L^2(\Omega^\epsilon)),$$

$$u_t^\epsilon \in C((0, T); H_{\partial\Omega}^1(\Omega^\epsilon)).$$

For the special geometry of our problem, we can use the following well-known lemma, due to C. Conca ([8]):

Lemma 5.1. There exists a positive constant C , independent of ϵ , such that

$$\|v\|_{\Omega^\epsilon} \leq C(\|\nabla v\|_{\Omega^\epsilon} + \epsilon^{\frac{1}{2}}\|v\|_{\partial F^\epsilon}), \quad (24)$$

for any $v \in H^1(\Omega^\epsilon)$, $v = 0$ on $\partial\Omega$.

The next proposition gives us some classical energy estimates for such a regular solution.

Proposition 5.2. For the system (1)-(5), the following classical parabolic estimates hold:

$$\sup_{t \in (0, T)} (\|u^\epsilon\|_{\Omega^\epsilon}^2 + \epsilon \|u^\epsilon\|_{\partial F^\epsilon}^2) + \|\nabla u^\epsilon\|_{\Omega^\epsilon, T} \leq C, \quad (25)$$

$$\|u_t^\epsilon\|_{\Omega^\epsilon, T}^2 + \epsilon \|u_t^\epsilon\|_{\partial F^\epsilon, T}^2 + \sup_{t \in (0, T)} \|u^\epsilon\|_{H_{\partial\Omega}^1(\Omega^\epsilon)}^2 \leq C. \quad (26)$$

Proof. The proof follows the standard steps for getting classical parabolic estimates (see, for instance, [12]) and, henceforth, can be omitted. In fact, using Lemma 4.1. and classical parabolic estimates, for such a regular solution one gets

Theorem 5.3. There exists an extension \tilde{u}^ϵ of the solution u^ϵ of problem (1)-(5) into Ω such that

$$\|\tilde{u}^\epsilon\|_{\Omega} + \|\nabla \tilde{u}^\epsilon\|_{\Omega, t} + \left\| \frac{\partial \tilde{u}^\epsilon}{\partial t} \right\|_{\Omega} + \left\| \frac{\partial \nabla \tilde{u}^\epsilon}{\partial t} \right\|_{\Omega, t} \leq C, \quad (27)$$

for any $t \leq T$. Here, C depends on the data and on T .

Proof. The proof follows immediately from the extension lemma and classical parabolic estimates.

Let us introduce now the vector $\mathbf{x}^\epsilon = \mathbf{c}^\epsilon \nabla \tilde{u}^\epsilon$. Recall that \mathbf{q} is the weak-* limit in $L^\infty(\Omega)$ of \mathbf{c}^ϵ .

Lemma 5.4. There exist two functions, $u \in \bar{V}$ (u will be the unique solution of the limit system (12)-(15)) and $\mathbf{x} \in \bar{H}$ such that, at least after the extraction of a subsequence, we have the following convergences:

$$\tilde{u}^e \xrightarrow{w} u \quad \text{weakly in } \bar{V} \text{ and strongly in } \bar{H}, \quad (28)$$

$$\frac{\partial \tilde{u}^e}{\partial t} \xrightarrow{w} \frac{\partial u}{\partial t} \quad \text{weakly in } \bar{H}, \quad (29)$$

$$\mathbf{c}^e \tilde{u}^e \xrightarrow{w} \mathbf{q} u \quad \text{weakly in } \bar{H}, \quad (30)$$

$$\mathbf{x}^e \xrightarrow{w} \mathbf{x} \quad \text{weakly in } \bar{H}, \quad (31)$$

$$\mathbf{e} \left(\frac{\partial u^e}{\partial t} \cdot \mathbf{j} \right)_{\partial F^e, T} \rightarrow \frac{|\partial F|}{|Y|} \left(\frac{\partial u}{\partial t} \cdot \mathbf{j} \right)_{\Omega, T} \quad \forall \mathbf{j} \in \bar{D}. \quad (32)$$

Proof of Lemma 5.4. The convergences (28) and (29) are direct consequences of the estimates given by Proposition 5.2. and Theorem 5.3.. The next one, (30), follows immediately from the fact that $\tilde{u}^e \rightarrow u$ strongly in \bar{H} and $\mathbf{c}^e \xrightarrow{*} \mathbf{q}$ weakly-* in $L^\infty(\Omega)$. Also, (31) follows from our a priori estimates. Indeed, we have $\|\mathbf{x}^e\|_{\Omega, T} \leq C$ and, hence, up to a sequence, there exists $\mathbf{x} \in \bar{H}$ such that $\mathbf{x}^e \xrightarrow{w} \mathbf{x}$ weakly in \bar{H} . It remains to prove (32). Let us consider a test function $\mathbf{j} \in \bar{D}$. It is easy to see that choosing $h = 1$ and taking $w^e = \tilde{u}^e \mathbf{j}_t$, from (23) we get

$$\mathbf{e} \int_{\partial F^e} u^e \mathbf{j}_t \, dS = \left\langle \mathbf{m}_1^e, \tilde{u}^e \mathbf{j}_t \Big|_{\Omega^e} \right\rangle \rightarrow \mathbf{m}_1 \int_{\Omega} u \mathbf{j}_t \, dx = \frac{|\partial F|}{|Y|} \int_{\Omega} u \mathbf{j}_t \, dx, \quad (33)$$

which, integrating in time and using Lebesgue's convergence theorem, gives exactly (32). This ends the proof of our lemma.

Now, let us come back to the proof of Theorem 3.1.. It remains only to obtain the limit equation (12) satisfied by u and \mathbf{x} . Let $\mathbf{j} \in \bar{D}$. We have:

$$\int_{\Omega^e} u^e \mathbf{j}_i dx = \int_{\Omega} \mathbf{c}^e \tilde{u}^e \mathbf{j}_i dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} u \mathbf{j}_i dx, \quad (34)$$

$$\int_{\Omega^e} \nabla u^e \nabla \mathbf{j} dx = \int_{\Omega} \mathbf{c}^e \nabla \tilde{u}^e \cdot \nabla \mathbf{j} dx \rightarrow \int_{\Omega} \mathbf{x} \cdot \nabla \mathbf{j} dx, \quad (35)$$

$$\mathbf{e} \int_{\partial F^e} \mathbf{g} \mathbf{j} d\mathbf{s} = \langle \mathbf{m}_1^e, \mathbf{g} \mathbf{j} \rangle \rightarrow \mathbf{m}_1 \int_{\Omega} \mathbf{g} \mathbf{j} dx \quad (36)$$

$$\int_{\Omega^e} \mathbf{f} \mathbf{j} dx = \int_{\Omega} \mathbf{c}^e \mathbf{f} \mathbf{j} dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} \mathbf{f} \mathbf{j} dx. \quad (37)$$

So, all the terms in (19) pass to the limit, as $\mathbf{e} \rightarrow 0$ and, therefore, we get

$$\begin{aligned} & \left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \left(u, \frac{\partial \mathbf{j}}{\partial t} \right)_{\Omega, T} + (\mathbf{x}, \nabla \mathbf{j})_{\Omega, T} = \\ & = \frac{|Y^*|}{|Y|} (f, \mathbf{j})_{\Omega, T} + \frac{|\partial F|}{|Y|} (g, \mathbf{j})_{\Omega, T} \quad \forall \mathbf{j} \in \overline{D}. \end{aligned} \quad (38)$$

But exactly like in the elliptical case (see [4]), we have

$$\mathbf{x}_i = q_{ij} \frac{\partial u}{\partial x_j}. \quad (39)$$

Finally, putting together (33)-(39) and having in mind that the solution of the macromodel is unique, the entire sequence of solutions of the microscopic model converges as necessary. So, we get (12). Since (13) and (14) are obviously satisfied, the proof of Theorem 3.1. for regular solutions is complete. As already mentioned, the proof in the general case comes easy, since it is enough, by density arguments, to perform a classical regularization of our data. This ends the proof of the theorem.

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