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ABOUT THE GENERALISATION OF THE KALUZA-KLEIN THEORY

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Abstract: The goal of this paper is to present the Kaluza-Klein theory. In the first part we will discuss the theory elaborated by Kaluza and Klein, in a Riemannian space with five dimensions, which unifies the gravitation with electromagnetism. The second part debates the generalization of this theory in a space with 4+n dimensions. This is a mathematical product between the Riemannian 4-dimension variety and the G/H n-dimensional homogenous space. In the last part we will propose a theory Kaluza-Klein like in the fiber bundle space with 4+n dimensions.

Every part is structured as follows: we will identify the metric tensor G for the gravitation and the potentials Yang-Mills; then we will deduct the equations of geodesics and the equations of the field.

1. The Kaluza-Klein theory

In the year 1916, Albert Einstein published "the generalized theory of gravity". In this theory is considered that, in the presence of the gravitational field, the 4-dimensional space is a pseudo Riemannian manifold M^4. In this space, a coordinate transformation has the form

$$x'^{\alpha} = x'^{\alpha}(x) \tag{1.1}$$

where $x = (x', x^2, x^3, x^4)$ and $a = 1,2,3,4$

From (1.1) results

$$dx'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\nu}} dx^{\nu} = a_{n}^{\cdot\alpha} dx^{\nu} \tag{1.2}$$

Because $a_{i}^{\cdot\alpha}$ is depending on *x*, this will be local coordinate transform. The line element for the 4-dimensional manifold M^4 is

$$ds^2 = G_{ab}\, dx^a\, dx^b \tag{1.3}$$

where G_{ab} is the metric tensor. In a pseudo Riemannian space the common derivative ∂_a is replaced with the covariant derivative

$$D_a = \partial_a \pm \Gamma_a \tag{1.4}$$

where the matrix elements $\Gamma_a = (\Gamma_{an}^{b})$ are connection coefficients. We can affirm that the presence of a gravitational field imposes the replacement of common derivative ∂_n with the covariant derivative D_n. In a local coordinate transformation, Γ_a is transformed

$$\Gamma_a^{'} = a_{\cdot a}^{n}\,(a\Gamma_n a^{-1} - (\partial_n a)a^{-1}) \tag{1.5}$$

where $\alpha = (a_{n}^{\cdot\alpha})$. From $D_n G_{ab} = 0$ results

$$\Gamma^{a}_{nb} = G^{ae}\frac{1}{2}\left(\partial_n G_{eb} + \partial_b G_{en} - \partial_e G_{bn}\right) \quad (1.6)$$

The motion equations on M^4 are the geodesics's equations.

$$\dot{u}^{a} + \Gamma^{a}_{bn} u^{b} u^{n} = 0 \quad (1.7)$$

The curvature tensor components are

$$R^{a}_{bne} = \partial_n \Gamma^{a}_{be} - \partial_e \Gamma^{a}_{bn} + \Gamma^{m}_{be}\Gamma^{a}_{mn} - \Gamma^{a}_{me}\Gamma^{m}_{bn} \quad (1.8)$$

From (1.8) we obtain the Ricci's tensor components

$$R^{n}_{bne} = \partial_n \Gamma^{n}_{be} - \partial_e \Gamma^{n}_{bn} + \Gamma^{m}_{be}\Gamma^{n}_{mn} - \Gamma^{n}_{me}\Gamma^{m}_{bn} \quad (1.9)$$

and the scalar curvature $$R = G^{be} R_{be} \quad (1.10)$$

The equations of the gravitational field are Einstein's equations:

$$R_{ab} - \frac{1}{2} G_{ab} R = \frac{8pG}{c^4} T_{ab} = J_{ab} \quad (1.11)$$

Here G is the gravitational constant and T_{ab} is the impulse-energy tensor for matter. These equations can be obtained from a variation principle by choosing $S = c^3 (16pG)^{-1} \int \sqrt{-G}\,(R + L_m)\, d^4 x$. The term J_{ab} is obtained from L_m which is the lagrangian "density" for matter.

In the year 1921, Kaluza suggested that the gravity and the electromagnetism can be unified in a pseudo Riemannian 5-dimensional space. Because the 5th dimension is not observable, Klein suggested that this is a circle S^1 with a small enough radius so it can't be experimentally detected. So, the 5 dimensional manifold is M^4 x S^1. In the Kaluza-Klein theory is postulated that the line element for the M^4 x S^1 space is

$$dS^2 = G_{ab}\, dx^a\, dx^b - \left(dy + kA_a(x) dx^a\right)^2 \quad (1.12)$$

where $x^5 = y$ is the 5th coordinate. The k constant is introduced in order that $kA_a(x)$ to be dimensionless. Also y has length dimension.

The coordinate (symmetry) transformations that leave the line element dS^2 invariant are

$$x'^{\acute{a}} = x'^{\acute{a}}(x) \quad (1.13)$$

$$y' = y + \boldsymbol{j}(x) \quad (1.14)$$

where $x = (x', x^2, x^3, x^4)$ and $\boldsymbol{a} = 1,2,3,4$. Under these transformations the M^4 line element $ds^2 = G_{ab}\, dx^a\, x^b$ is invariant and $A_a(x)$ it transforms

$$A'_a(x') = d^{n}_{\cdot a}\left(A_n(x) - \frac{1}{k}\frac{\partial \boldsymbol{j}(x)}{\partial x^n}\right) \quad (1.15)$$

For a pseudo Riemannian 5-dimensional manifold the line element is

$$ds^2 = g_{ab}\, dx^a dx^b \quad a,b = 1,2,3,4,5 \quad (1.16)$$

and is invariant under the coordinate (symmetry) transformations

$$x'^{a} = x'^{a}(x)\,,\ x = (x', x^2, x^3, x^4, x^5) \tag{1.17}$$

In the basis $\{dx^a, dy\}$ for one-forms and in the dual basis $\left\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y}\right\}$ in tangent space, the line element (1.12) is :

$$dS^2 = (G_{ab} - k^2 A_a A_b)dx^a dx^b - 2kA_a dx^a dy - g_{55}dydy \tag{1.18}$$

In this basis, the metric tensor components are:

$$g_{ab} = G_{ab} - k^2 A_a A_b\,,\ g_{55} = 1\,,\ g_{a5} = g_{5a} = -kA_a \tag{1.19}$$

The line element dS^2 given by (1.18) is invariant under transformations (1.13) and (1.14) which are obtained by the symmetry "breaking" (1.17). In this case, the $g_{\alpha 5}$ components of the metric tensor act as a vectorial gauge field, A_α which is considered to be the electromagnetic field potentials.

Observation: Let $\Psi(x)$ be a scalar field, which transforms

$$\Psi'(x) = a(x)\Psi(x)\,,\ a(x) = e^{j(x)} \tag{1.20}$$

We have: $D'_a \Psi'(x) = \partial_a \Psi'(x) + kA'_a(x)\Psi'(x) = a(x)(\partial_a \Psi(x) + kA_a \Psi(x)) = a(x)D_a \Psi(x)$ (1.21)

which means that

$$D_a = \partial_a + kA_a \tag{1.22}$$

acts like a covariant derivative under the transformations (1.20).

If for one-forms we choose the basis

$$dX^a = dx^a\,,\ dX^5 = dy + kA_a dx^a \tag{1.23}$$

then the dual basis in tangent space is:

$$e_a = \frac{\partial}{\partial x^a} - kA_a \frac{\partial}{\partial y}\,,\ e_5 = \frac{\partial}{\partial y} \tag{1.24}$$

We have:

$$[e_a, e_b] = C^c_{ab} e_c\,,\ a,b,c = 1,2,3,4,5 \tag{1.25}$$

$$[e_a, e_b] = -kF_{ab}\frac{\partial}{\partial y}\,,\ [e_a, e_5] = 0\,,\ [e_5, e_5] = 0 \tag{1.26}$$

$$\Gamma_{bcd} = \frac{1}{2}\left(e_d(G_{bc}) + e_c(G_{bd}) - e_b(G_{dc}) + C_{dcb} + C_{bcd} + C_{bdc}\right) \tag{1.27}$$

$$G = \begin{pmatrix} G_{ab} & 0 \\ 0 & -g_{55} \end{pmatrix} \tag{1.28}$$

In this basis the line element is:

$$dS^2 = G_{ab}dx^a dx^b - g_{55}dX^5 dX^5 \tag{1.29}$$

Calculating the connection coefficients we get:

$$\Gamma^a_{bn} = G^{ae}\Gamma_{ebn}\,,\ \Gamma^5_{bn} = \frac{1}{2}kF_{bn}\,,\ \Gamma^a_{b5} = \frac{1}{2}kF^a_b\,,\ \Gamma^5_{b5} = 0 \tag{1.30}$$

The geodesics's equations au

$$\dot{u}^a + \Gamma^a_{ab} u^a u^b = 0 \quad a,b = 1,2,3,4,5$$
$$\dot{U}^5 + \Gamma^5_{ab} u^a u^b = 0 \quad \alpha = 1,2,3,4 \tag{1.31}$$

By replacing (1.30) in (1.31) we get

$$\dot{U}^5 = 0 \tag{1.32}$$

$$G_{ab}\dot{u}^b + \Gamma_{abn} u^b u^n = g_{55} U^5 k F_{an} u^n \tag{1.33}$$

By choosing $g_{55} = 1$, $U^5 = 1$ and $k = \frac{q}{m_0 c^2}$ we get

$$\dot{u}^b + \Gamma^b_{an} u^a u^n = \frac{q}{m_0 c^2} F_n^{\ b} u^n \tag{1.34}$$

where $\Gamma^b_{an} = G^{be}\Gamma_{ean}$, $\Gamma_{ean} = \frac{1}{2}\left(\frac{\partial G_{ea}}{\partial x^n} + \frac{\partial G_{en}}{\partial x^a} - \frac{\partial G_{an}}{\partial x^e}\right)$ for the gravitational field (1.35)

and $F_n^{\ b} = G^{be} F_{en}$, $F_{en} = \frac{\partial A_n}{\partial x^e} - \frac{\partial A_e}{\partial x^n}$ for the electromagnetic field (1.36)

(1.34) is the motion equation for a particle with the mass at rest m_0 and the charge q in the gravitation and electromagnetic field.

The field equations are Einstein equations in the space $M^4 x S^1$:

$$R_{ab} - \frac{1}{2} G_{ab} R = J_{ab}, \ a,b = 1,2,3,4,5 \tag{1.37}$$

After calculating we get:

$$R^4_{ab} - \frac{1}{2} G_{ab} R^4 - \frac{8\pi G}{c^4} T_{ab} = J_{ab} \tag{1.38}$$

where 4 is an index that shows that R^4_{ab} is the Ricci's tensor on M^4, R^4 is the curvature scalar on M^4 and

$$T_{ab} = \varepsilon_0 \left[\frac{1}{4} G_{ab} F^n_e F^e_n - \frac{1}{2}\left(F^e_b F_{ae} + F^e_a F_{be}\right)\right] \tag{1.39}$$

is the energy-impulse tensor for the electromagnetic field. In the chosen basis $G_{a5} = 0$ and therefore $R_{a5} = J_{a5}$ from which results

$$\partial_n F^n_a + \Gamma^n_{en} F^e_a - \Gamma^n_{ea} F^e_n = 2k^{-1} J_{a5} = J_a \tag{1.40}$$

Using the covariant derivative $D_n = \partial_n \pm \Gamma_n$ equation (1.40) becomes

$$D_n F^n_a = J_a \tag{1.41}$$

Also, is satisfied the identity

$$D_a F_{bn} + D_b F_{na} + D_n F_{ab} = 0 \tag{1.42}$$

The equations (1.37) are the equations of the gravitational field for a pseudo Riemannian five dimensional manifold. By symmetry breaking, the equation (1.37) becomes (1.38) and (1.41) which are equations for the gravitational field and for electromagnetic field on the M^4 manifold.

The field equations can be obtained from a variation principle by taken the action to be the gravitational action in the five dimensional spaces:

$$S_5 = \frac{c^3}{16pG \cdot L}\int \sqrt{g}(R + L_m)d^5x \qquad (1.43)$$

After integrating over y results $S_4 = \frac{c^3}{16pG}\int \sqrt{-G}(R + L_m)d^4x$ (1.44)

where $\sqrt{-G} = \sqrt{-\det(G_{ab})}$, $R = R^4 - \frac{1}{4}k^2F^2$ cu $F^2 = F_e^n F_n^e$ and $k^2 = \frac{16pe_0 G}{c^4}$.

2. The Kaluza-Klein theory in a M^4 x S^n space

On the space S^n transitively acts a group Lie G. This means that for two arbitrary points from S^n we have

$$y' = f(g, y) \quad y', y \in S^n \qquad (2.1)$$

for $g \in G$. Let $y_0 \in S_n$ and *H* the set of all *G* elements that let y_0 fix:

$$H = \{h \in G \mid y_0 = f(h, y_0)\} \qquad (2.2)$$

H is an isotropy subgroup of *G*. For every $g \in G$ we can define an equivalence class

$$[g] = \{g' \in G \mid g' = gh \,,\, \forall h \in H\} \qquad (2.3)$$

The manifold of all distinct equivalence classes is noted *G/H*. Let $y_0 \in S^n$ fixed. Any point $y \in S^n$ can be obtained: $y = f(g, y_0)$, $g \in G$. But $f(gh, y_0) = f(g, f(h, y_0)) = f(g, y_0)$ for any $h \in H$. Results that there is a biunique correspondence between the equivalence classes [g] and the points $y \in S^n$. We will identify the space S^n with the coset space *G/H* so

$$\dim G/H = \dim G - \dim H \qquad (2.4)$$

For a Kaluza-Klein theory of supergravity the number of supplementary dimensions must be maximum seven, therefore for the space S^n we have n≤ 7. If $G = SU(3) \times SU(2) \times U(1)$ then $H = U(2) \times U(1)$ and dim *G/H*=7.

We choose the coordinate transformations on M4 x S^n

$$x'^a = x'^a(x) \,,\, a = 1,2,3,4 \,,\, x = (x^1, x^2, x^3, x^4) \qquad (2.5)$$

$$y'^k = y'^k(w(x), y) \,,\, k = 1,2,\ldots,n \,,\, y = (y^1, y^2, \ldots, y^n) \qquad (2.6)$$

ω(x) are the parameters of the *G* group which acts on S^n. The symmetry is broken because the transformations (2.5) and (2.6) aren't like

$$x'^a = x'^a(x) \,,\, a = 1,2,\ldots,4+n \qquad (2.7)$$

$$x = (x^1, x^2, x^3, x^4, x^{4+1}, \ldots x^{4+n}) \,, \quad x^{4+1} = y^1, x^{4+2} = y^2, \ldots, x^{4+n} = y^n$$

The line element Kaluza-Klein, generalized for a space with cu 4+n dimensions, is

$$dS^2 = G_{ab}dx^a dx^b - g_{rk}dX^r dX^k \qquad (2.8)$$

where we chose the basis one-forms

$$dX^a = dx^a \,,\, dX^r = dy^r + kN_a^r dx^a \qquad (2.9)$$

k is a constant introduced in order that kN_a^r be dimensionless.

The basis vectors from the tangent space are

$$e_a = \frac{\partial}{\partial x^a} - kN_a^r \frac{\partial}{\partial y^r},\ e_k = \frac{\partial}{\partial y^k} = \partial_k \qquad (2.10)$$

G_{ab} is depending on x, g_{sk} is depending on y, and N_a^s is depending on x and y.

Let dS^2 be invariant under transformations (2.5) and (2.6). It is seen that $ds^2 = G_{ab}dx^a dx^b$ is invariant under these transformations. Because

$$g'_{rk}(y') = \frac{\partial y^n}{\partial y'^r}\frac{\partial y^s}{\partial y'^k} g_{ns}(y) = a^n_{\cdot r}(y) a^s_{\cdot k}(y) g_{ns}(y) \qquad (2.11)$$

results $dX'^r = \frac{\partial y'^r}{\partial y^s} dX^s = a^{\cdot r}_s(y) dX^s$ therefore $dy'^r + kN'^r_a(x', y')dx'^a = a^{\cdot r}_s(y)(dy^s + kN^s_a(x,y)dx^a)$.

From here results

$$N'^s_a = a^n_{\cdot a}\left(a^{\cdot s}_r N^r_n - \frac{1}{k}\frac{\partial y'^s}{\partial x^n} \right) \qquad (2.12)$$

Let the coordinate transformations

$$x'^a = x^a \qquad (2.13)$$

$$y'^k = y'^k(w(x), y) \qquad (2.14)$$

For $w(x)$ infinitesimal the transformation (2.14) can be written

$$y'^r = y^r + \frac{\partial y'^r}{\partial w^{n_0}}\Big|_{w^{n_0}=0} \cdot dw^{n_0} \qquad (2.15)$$

Noting $\frac{\partial y'^r}{\partial w^{n_0}}\Big|_{w^{n_0}=0} = K^r_{n_0}(y) = K^r_{n_0}$, we have

$$y'^r = y^r + K^r_{n_0} dw^{n_0} \qquad (2.16)$$

The operators $X_{n_0} = K^r_{n_0}(y)\partial_r$ (2.17)

are called the generators of the transformations group *G* which acts on S^n, transformations given by (2.14). It can be showed that

$$\left[X_{n_0}, X_{m_0}\right] = C^{r_0}_{n_0 m_0} X_{r_0} \qquad (2.18)$$

where $C^{r_0}_{n_0 m_0}$ are the structure constants of the group. From (2.18) results

$$K^r_{n_0}\partial_r K^s_{l_0} - K^r_{l_0}\partial_r K^s_{n_0} = C^{i_0}_{n_0 l_0} K^s_{i_0} \qquad (2.19)$$

If the transformation (2.14) is an isommetry $K^n_{i_0}$ will be solutions of the Killing's equations

$$K^n_{i_0}\partial_n g_{rk} + g_{kn}\partial_r K^n_{i_0} + g_{rn}\partial_k K^n_{i_0} = 0 \qquad (2.20)$$

$K^n_{i_0}$ are called Killing's vectors. In particular, the equations (2.20) are satisfied if we choose

$$g^{rs}(y) = K^{r}_{i_0}(y) K^{s}_{i_0}(y) \cdot L^{-2} \tag{2.21}$$

Here L is a constant with length dimension introduced in order that g^{rs} be dimensionless. We choose

$$N^{r}_{a}(x,y) = A^{s_0}_{a}(x) K^{r}_{s_0}(y) L^{-1} \tag{2.22}$$

in a transformation (2.16) the fields $A^{n_0}_{n}(x)$ transforms

$$A'^{n_0}_{n} = A^{n_0}_{n} + C^{n_0}_{s_0 r_0} A^{s_0}_{n} \boldsymbol{dw}^{r_0} - Lk^{-1} \partial_n \left(\boldsymbol{dw}^{n_0} \right) \tag{2.23}$$

This is an infinitesimal gauge transformation for the gauge fields $A^{n_0}_{n}$ (x).

The connection coefficients are obtained from

$$\Gamma_{bcd} = \frac{1}{2} \left(e_d \left(G_{bc} \right) + e_c \left(G_{bd} \right) - e_b \left(G_{dc} \right) + C_{dcb} + C_{bcd} + C_{bdc} \right) \tag{2.24}$$

where:

$$G = \begin{pmatrix} G_{ab} & 0 \\ 0 & -g_{rk} \end{pmatrix} \tag{2.25}$$

$$\left[e_a , e_b \right] = C^{c}_{ab} e_c , \; a,b,c,=1,2,...,4+n \tag{2.26}$$

$$\left[e_a , e_b \right] = -k F^{n_0}_{ab} K^{r}_{n_0} \partial_r = C^{r}_{ab} \partial_r , \; \left[e_a , e_s \right] = k A^{n_0}_{a} \partial_s K^{r}_{n_0} \partial_r = C^{r}_{as} \partial_r = -C^{r}_{sa} \partial_r , \quad \left[e_r , e_s \right] = 0 \tag{2.27}$$

$$F^{n\, n_0}_{b} = G^{na} F^{n_0}_{ab} \;, \; F^{n_0}_{ab} = \partial_a A^{n_0}_{n} - \partial_n A^{n_0}_{a} + k L^{-1} A^{i_0}_{n} A^{s_0}_{a} C^{n_0}_{i_0 s_0} \tag{2.28}$$

After calculating results:

$$\begin{gathered} \Gamma^{a}_{br} = \Gamma^{a}_{rb} = G^{ae} \Gamma_{ebr} = -\frac{1}{2} k L^{-1} F^{a\, n_0}_{b} K^{s}_{n_0} g_{sr} \;, \quad \Gamma^{r}_{ab} = g^{rs} \Gamma_{sab} = -\frac{1}{2} k L^{-1} F^{n_0}_{ab} K^{r}_{n_0} \\ \Gamma^{a}_{bn} = G^{ae} \Gamma_{ebn} \;, \quad \Gamma^{r}_{sa} = g^{rn} \Gamma_{nsa} = k L^{-1} A^{n_0}_{a} \partial_s K^{r}_{n_0} , \quad \Gamma^{r}_{as} = g^{rn} \Gamma_{nas} = 0 \\ \Gamma^{r}_{sk} = g^{rn} \frac{1}{2} \left(\partial_k g_{ns} + \partial_s g_{nk} - \partial_n g_{sk} \right) \;, \quad \Gamma^{a}_{rs} = 0 \end{gathered} \tag{2.29}$$

The geodesics's equations are

$$\dot{u}^{b} + \Gamma^{b}_{na} u^{n} u^{a} + \Gamma^{b}_{rs} U^{r} U^{s} + 2 \Gamma^{b}_{nr} U^{r} u^{n} = 0 \tag{2.30}$$

$$\dot{U}^{k} + \Gamma^{k}_{ab} u^{a} u^{b} + \Gamma^{k}_{as} U^{s} u^{a} + \Gamma^{k}_{sa} U^{s} u^{a} + \Gamma^{k}_{rs} U^{r} U^{s} = 0 \tag{2.31}$$

Using (2.29) in (2.30) and (2.31) this becomes

$$\dot{U}^{k} + \Gamma^{k}_{rs} U^{r} U^{s} + k L^{-1} A^{n_0}_{n} \partial_s K^{k}_{n_0} U^{s} u^{n} = 0 \tag{2.32}$$

$$\dot{u}^{b} + \Gamma^{b}_{an} u^{a} u^{n} = k L^{-1} F^{b\, n_0}_{n} K^{r}_{n_0} P_r u^{n} \tag{2.33}$$

Observation: the operator ∂_r corresponds to the medium value $L^{-1}P_r$. Results that the factor $K^{r}_{n_0} P_r L^{-1}$ is the medium value of the operator $K^{r}_{n_0} \partial_r = X_{n_0}$. Assuming that qX_{n_0} is the "field's charge" operator, then its medium value $qK^{r}_{n_0} P_r L^{-1}$ is noted Q_{n_0}. Because $\dot{P}_r - \Gamma^{s}_{rk} P_s U^{k} - \Gamma^{s}_{ra} P_s u^{a} = 0,$ the condition $\dot{Q}_{n_0} = 0$ implies $\dot{K}^{r}_{n_0} + \Gamma^{r}_{sk} K^{s}_{n_0} U^{k} + \Gamma^{r}_{sa} K^{s}_{n_0} u^{a} = 0$. With this (2.33) becomes

$$\dot{u}^b + \Gamma^b_{an} u^a u^n = \frac{Q_{n_0}}{m_0 c^2} F_n^{\ b n_0} u^n \qquad (2.34)$$

Observation: We define the Yang-Mills field

$$Y_a(x,y) = A_a^{n_0} X_{n_0} \qquad (2.35)$$

If $X^{\ a}_{n_0 b}$ are the elements of the generators representation matrix X_{n_0} then $Y^{\ a}_{ab}(x,y) = A_a^{n_0}(x) X^{\ a}_{n_0 b}(y)$. Let $\Psi^c(x,y)$ the components of a vectorial field in the representation space. They transform:

$$\Psi'^c(x,y') = a_b^{\cdot c}(x,y)\Psi^b(x,y), \quad a = \left(a_b^{\cdot c}\right), \quad a = e^{w^{n_0} X_{n_0}} \qquad (2.36)$$

The field transformation $Y^{\ a}_{ab}$ is obtained from (2.23)

$$Y_a^{'} = aY_a a^{-1} - Lk^{-1}(\partial_a a)a^{-1} \qquad (2.37)$$

The covariant derivative in the representation space is

$$D_x = \partial_a \pm gY_a \ \text{ and } \ D_n \Psi^a = \partial_n \Psi^a + gY^{\ a}_{n\,b}\Psi^b \qquad (2.38)$$

Where $g = kL^{-1} = 2\pi q(hc)^{-1}$ so gY_a has a dimension of (length)$^{-1}$.

The curvature tensor components for $M^4 \, x \, S^n$ are

$$R^e_{cab} = \partial_a \Gamma^e_{cb} - \partial_b \Gamma^e_{ca} + \Gamma^e_{ad}\Gamma^d_{cb} - \Gamma^e_{bd}\Gamma^d_{ca} + C^d_{ba}\Gamma^e_{cd}, \quad a,b=1,2,...,4+\mathrm{n} \qquad (2.39)$$

From (2.39) we obtain the components of the Ricci's tensor $R_{cb} = R^a_{cab}$ which gives R_{ab} and R_{ks}. The curvature scalar is $R = G^{ab}R_{ab} - g^{ks}R_{ks} = R^4 + R_{G/H} - \frac{1}{4}k^2F^2$. Here R^4 is the scalar curvature of M^4, $R_{G/H}$ the scalar curvature for *G/H* and $F^2 = g_{ij}K^i_{n_0}K^j_{m_0}F_e^{n\,n_0}F_n^{e\,m_0}$.

The field equations of M^4 can be obtained from a variation principle. We choose the action on $\mathrm{M}^4 \times \mathrm{S}^n$

$$S_{4+n} = \frac{c^3}{16\pi G V^u} \int_{M^4} \sqrt{-G}\, d^4x \int_{S^n} \sqrt{-g}\,(R + L_m)\, d^n x \qquad (2.40)$$

where $V^n = \int_{S^n} \sqrt{-g}\, d^n x$, $\sqrt{-G} = \sqrt{-\det(G_{ab})}$, $\sqrt{-g} = \sqrt{-\det(g_{rk})}$

After that, we integrate on S^n and results:

$$S_4 = \frac{c^3}{16\pi G} \int_{M^4} \sqrt{-G}\left(R^4 + \Lambda + L_m\right) d^4x - \frac{1}{4}\frac{\varepsilon_0}{c} \int_{M^4} \sqrt{-G}\, F_e^{nn_0} F_n^{en_0}\, d^4x \qquad (2.41)$$

where $\int_{S^n} \sqrt{-g}\, R_{G/H}\, d^n x = V^n \Lambda$. The field equations obtained from S_4 are:

$$R^4_{ab} - \frac{1}{2}G_{ab}\left(R^4 + \Lambda\right) - \frac{8\pi G}{c^4}T_{ab} = J_{ab} \qquad (2.42)$$

$$\partial_n F_a^{n\,n_0} + \Gamma^n_{en} F_a^{e\,n_0} - \Gamma^e_{an} F_e^{nn_0} = J_a^{n_0} \qquad (2.43)$$

where $T_{ab} = \varepsilon_0 \left[\frac{1}{4}G_{ab} F_e^{nn_0} F_n^{en_0} - \frac{1}{2}\left(F_a^{nn_0} F_{ba}^{n_0} + F_b^{nn_0} F_{an}^{n_0}\right)\right]$ is the energy-impulse tensor for the field $F_a^{n\,n_0}$.

3. The Kaluza-Klein theory in a fibre bundle space

We will consider a fibre bundle space for which the basis space is the differential variety M^4, and the typical fibre F^n is the set of all vectorial fields $\Psi(x)=\left\{\Psi^k(x)\right\}$ k=1,2,...,n. In every point $x=\left(x^1,x^2,x^3,x^4\right)$ of the manifold M^4 we have a fibre F_x^n, which is a vectorial space. A vector in this space is $\Psi(x)=\left\{\Psi^k(x)\right\}$ for *x* fixed. On the typical fibre F^n acts a transformation group *G*:

$$\Psi'^k(x)=a_r^{\cdot k}(x)\Psi^r(x) \tag{3.1}$$

Noting *E* the total space, the coordinates on E are $x^1,x^2,x^3,x^4,y^1,y^2,\dots,y^n$. A coordinate transformation on *E* is

$$x'^a=x'^a(x) \tag{3.2}$$

$$y'^k=a_r^{\cdot k}(x)y^r \tag{3.3}$$

If e_k is a base for F^n, then a vector from F^n is written:

$$\Psi(x)=e_k\Psi^k(x) \tag{3.4}$$

The coordinates y^k are defined for each fibre F_x^n. The variation of y^k in the fibre F_x^n is dy^k. When we pass from a fibre F_x^n to another fibre $F_{x'}^n$ the variation of y^k is $kN_a^k dx^a$ where $dx^a=x'^a-x^a$ and kN_a^k are the connection coefficients which are depending on *x* and y. The total variation of y^k is

$$Dy^k=dy^k+kN_a^k(x,y)dx^a \tag{3.5}$$

The “position” vector in F^n can be written $y=e_k y^k$ and the variation of this vector is $Dy=e_k Dy^k$. The corresponding variation on M^4 will be $dx=e_a dx^a$. An infinitesimal movement on *E* is

$$dX=dx+Dy=e_a dx^a+e_k Dy^k \tag{3.6}$$

The line element on *E* will be

$$dS^2=G_{ab}dx^a dx^b-g_{rk}Dy^r Dy^k \tag{3.7}$$

where $g(e_a,e_b)=G_{ab}$, $g(e_a,e_k)=G_{ak}=0$ and $g(e_r,e_k)=-g_{rk}$.

Under the transformations (3.2) and (3.3), the line element of M^4, $ds^2=G_{ab}dx^a dx^b$ is invariant. Results $g_{rk}Dy^r Dy^k$ must be invariant under these transformations. The invariance condition $g'_{rk}Dy'^r Dy'^k=g_{rk}Dy^r Dy^k$ and the transformation conditions $g'_{rk}=a_{\cdot r}^{s}a_{\cdot k}^{n}g_{sn}$ give $a_{\cdot r}^{s}Dy'^r=Dy^s$ from which results

$$kN'^r_a=a_{\cdot a}^{n}\left(a_s^{\cdot r}kN_n^s-\frac{\partial a_s^{\cdot r}}{\partial x^n}y^s\right) \tag{3.8}$$

Let the infinitesimal transformations

$$x'^a=x^a \tag{3.9}$$

$$y'^k=(\delta_r^k+\omega_r^{\cdot k}(x))y^r \tag{3.10}$$

If $w_r^{\cdot k} = q_{rn_0}^{\cdot k} w^{n_0}(x)$ then

$$y'^k = y^k + q_{rn_0}^{\cdot k} w^{n_0}(x) y^r \tag{3.11}$$

From (3.11) we see that the Killing's vectors are $K_{n_0}^r(y) = q_{sn_0}^{\cdot r} y^s$ and the (3.3) transformations generators are $X_{n_0} = K_{n_0}^r \partial_r = q_{sn_0}^{\cdot r} y^s \partial_r$. We choose $kN_a^r(x,y) = kL^{-1} A_a^{n_0}(x) K_{n_0}^r(y) = kL^{-1} A_a^{n_0}(x) q_{n_0 s}^r y^s$.

Observation: a) The Yang-Mills is $Y_a = A_a^{n_0} X_{n_0} = LN_a^r \partial_r = A_a^{n_0} q_{n_0 s}^r y^s \partial_r$. Therefore exists a linear connection $K_{as}^r(x) = gA_a^{n_0}(x) q_{n_0 s}^r$. From (3.8) with the notations $K_a = (K_{as}^r)$ and $a = (a_s^{\cdot r})$ results

$$K'_a = a_{\cdot a}^n \left(aK_n a^{-1} - (\partial_n a) a^{-1}\right) \tag{3.12}$$

The covariant derivative is $D_a = \partial_a \pm K_a$ and $D_a \Psi^r(x) = \partial_a \Psi^r(x) + K_{as}^r(x) \Psi^s(x)$ .If for the generators $y^s \partial_r$ we choose the representation $(y^s \partial_r)_i^j = \boldsymbol{d}_i^s \boldsymbol{d}_r^j$ then $gY_{a\,i}^{\,j} = K_{a\,i}^{\,j}(x)$.

b) If the field $\Psi^k(x)$ is a self-interacting field, then from the Higgs mechanism can be obtained at rest masses for the field quanta $A_n^{n_0}$.

Forwards the theory develops like in the second part but y^r are coordinates in "states space" (F^n) and $g_{rk} = \boldsymbol{d}_{rk}$. From a physical point of view, the states space it isn't a space like $x^4 = ct$ it isn't a space. We will consider that $y^r = \boldsymbol{l}\, y_0^r$ where $\boldsymbol{l}$ has a length dimension and y_0^r is dimensionless.

The connection coefficients will be

$$\Gamma_{bn}^a = G^{ae} \Gamma_{ebn} \,, \Gamma_{sa}^r = K_{as}^r \,, \quad \Gamma_{br}^a = \Gamma_{rb}^a = -\frac{1}{2} kL^{-1} F_b^{\,a\,n_0} K_{n_0}^s g_{sr} \tag{3.13}$$

After calculating, the geodesics equations become

$$\dot{u}^b + \Gamma_{an}^b u^a u^n = \frac{Q_{n_0}}{m_0 c^2} F_n^{\,b\,n_0} u^n \tag{3.14}$$

$$\dot{U}^r + K_{ns}^r U^s u^n = 0 \tag{3.15}$$

Using (3.13) the curvature scalar is $R = R^4 - \frac{1}{4} k^2 L^{-2} F^2$. The field equations on M^4 have the same form as in (2.42) and (2.43) but Λ is zero.

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