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APPLYING q-LAGUERRE POLYNOMIALS TO THE DERIVATION OF q-DEFORMED ENERGIES OF OSCILLATOR AND COULOMB SYSTEMS

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Abstract. The q-deformed radial Schrödinger equation written down before [E. Papp, Phys. Rev. A 52, 101 (1995)] is analyzed in terms of products between q-Laguerre polynomials and q-exponential functions. For this purpose Coulomb and harmonic oscillator potentials are considered. Eigenvalue equations are able to be handled quasiclassically, such that now the accuracy gets improved for larger values of the quantum number $L = l + (N-2)/2$. Here l stands for the quantum number of the angular momentum, whereas N denotes the number of space dimensions. This results in the application of an eigenvalue recipe proposed before [G. Chan, R. Finkelstein and V. Oganesyan, J. Math. Phys. 38 2135 (1997)]. Proceeding in this way we are able to establish quasiclassical alternatives to exact q-deformed energies obtained before by virtue of operator methods, but our results become exact ones for the ground-state.

Key words: q-difference equations, q-Laguerre polynomials, q-deformed oscillator and Coulomb systems.

1. INTRODUCTION

The description of the Coulomb-problem [1, 2] and of the harmonic oscillator [3, 4, 5, 6, 7] on the N-dimensional noncommutative Euclidean space (NCES) has received much interest. For this purpose Schrödinger-equations with $SO_q(N)$-deformed Laplace-operators have been treated with the help of operator methods [3], which amounts to proceed in terms of q-analogs of creation and annihilation operators [4, 5, 6]. The pertinent deformation parameter is denoted by q, such that $0 < q \leq 1$. The related q-deformation of the angular momentum has also been done [1, 2, 7]. In addition, the harmonic oscillator has been solved by invoking superpositions of q-exponential functions [4, 5]. Proofs have been given that energy-formulae established in this manner fulfill the q-deformed version of the Coulomb-oscillator duality [8]. For this purpose we found it suitable to rescale the energy of the q-deformed oscillator (see Eq. (2.36) in Ref. 4) by a $1/q^{N/2}$

factor. In contradistinction, the q-deformed Coulomb-energy written down before [1] is preserved as it stands. However, in order to open the way for several applications we have to derive explicit results concerning the wavefunctions, *i.e.*, to proceed in as a close connection with the classical description as possible. We then have to solve the q-deformed radial Schrödinger-equations [4, 9], such as given *e.g.*, by (34)–(36) in Ref. 10, now by considering products between q-Laguerre polynomials and q-exponential functions [11]. Such equations can be viewed as byproducts of the radial reduction of the covariant $SO_q(N)$-covariant derivative [1, 4, 9]. Note that q-Laguerre polynomials have been used before [12], but this has been done just for the usual Schrödinger-equation supplemented by an extra q-deformation of the angular-momentum. However, these latter issues are quite useful, which concerns especially quasiclassical "recipes" for computing the energy (see Eq. (6.1) in Ref. 12). Accordingly, we are led to combine the eigenvalue-equation with a limit like $r \to r_0$, such that the potential vanishes at $r = r_0$.

Our study is also motivated by recent advances concerning the description of electrons on a two-dimensional noncommutative plane threaded by a transversal and homogeneous magnetic field. This results in a q-deformed harmonic oscillator which is relevant to the quantum Hall-effect [13] and to an updated description of magnetic properties [14, 15]. In these latter applications the concrete implementation of the noncommutative geometry is different from the present one, but the corresponding deformation-parameters are interrelated [16]. Moreover, the present study is of a general theoretical interest, as there are links with generalized formulations of thermodynamics [17] as well as with the description of the Coulomb-problem [18] and oscillator systems [18, 19, 20] on discrete spaces. In addition, there are inter-corrections with other research fields like q-uncertainty relations [21] or the q-deformed phase-space [22].

This paper is organized as follows. The q-Laguerre polynomials and the general form of radial q-wavefunctions are presented in Section 2. Section 3 deals with the description of the Coulomb-problem in terms of q-Laguerre polynomials, now by applying the quasiclassical $r \to \infty$ limit. The harmonic oscillator is discussed in a close connection with the duality transformation in Section 4. Concluding remarks are finally made in Section 5. Units for which $\hbar^2 = 2m = e^2 = 1$ are used.

2. PRELIMINARIES AND NOTATIONS

Choosing "*f*-representation" *i.e.* (34) in Ref. 10, one gets faced with the separated q-deformed radial Schrödinger-equation

$$H_q f_l(r;q) = \left(-\frac{\mu^2}{(q+1)^2} \left(q^{2L+1} \partial_r^{(q)^2} + [[2L+1]]_q \frac{1}{r} \partial_r^{(q)} \right) + V(r) \right) f_l(r;q) = \\ = E_q f_l(r;q) \tag{1}$$

where $\mu = 1 + q^{2-N}$ and $L = l + (N-2)/2$. The usual q deformed radial wavefunction is given by $\psi_l(r;q) = r^l f_l(r;q)$, where $l = 0,1,2,\ldots,$ denotes the quantum number of the angular momentum. Within the classical $q = 1$ limit one has $f_l(r;1) \in \{L_2(0,\infty), r^{2L+1} dr\}$, whereas $\psi_l(r;1) \in \{L_2(0,\infty), r^{N-1} dr\}$.

The present Jackson-derivative [11] proceeds as

$$\partial_r^{(q)} f(r) \equiv \frac{\mathrm{d}_q f(r)}{\mathrm{d}_q r} = \frac{f(qr) - f(r)}{r(q-1)} , \tag{2}$$

so that

$$\partial_r^{(q)} r^n = [[n]]_q \, r^{n-1}, \tag{3}$$

where

$$[[n]]_q \equiv \frac{q^n - 1}{q-1} \equiv q^{\frac{n-1}{2}} [n]_{\sqrt{q}} . \tag{4}$$

It is also clear that now the Leibniz rule gets modified, so that

$$\partial_r^{(q)^2} \left(f(r) g(r) \right) = g(r) \partial_r^{(q)^2} f(r) + \frac{q+1}{q} \partial_r^{(q)} f(qr) \partial_r^{(q)} g(r) + f\left(q^2 r\right) \partial_r^{(q)^2} g(r). \tag{5}$$

On the other hand the classical radial wavefunctions for the radial Schrödinger-equation with the Coulomb $(V_1 = -Z/r)$ and harmonic oscillator $(V_2 = \omega_0^2 r^2)$ potentials are given, up to normalization constants, by [23, 24]

$$f_l^{(1)}(r;1) = \exp\left(-\frac{Zr}{2d_1} \right) L_n^{(2L)} \left(\frac{Zr}{d_1} \right), \tag{6}$$

and

$$f_l^{(2)}(r;1) = \exp\left(-\frac{\omega_0 r^2}{2} \right) L_n^{(L)} \left(\omega_0 r^2 \right), \tag{7}$$

respectively. The principal quantum numbers are denoted by $d_1 = L + n_r + 1/2$ and $d_2 = L + 2n_r + 1$, respectively, whereas $n = n_r = 0, 1, 2, \ldots$ is the common radial quantum number. Probing duality attributes it is suitable to insert $L = L_1 = l_1 + n_r + (N_1 - 1)/2$ and $L = L_2 = l_2 + 2n_r + N_2/2$, respectively, in which case $(N_2-2)/(N_1-2)=l_2/l_1=2$ [19, 20]. Using $\mu_j=1+q_j^{2-N_j}\,(j,2)$, one sees immediately that $\mu_1=\mu_2=\mu$ if $q_1=q^2$ and $q_2=q$. It is also clear that the classical energies $E_1^{(1)}=-Z^2/4d_1^2$ and $E_1^{(2)}=2\omega_0 d_2$ should be reproduced as soon as $q\to 1$.

Dealing with the q-deformation of (6) and (7) we shall resort to the q-Laguerre polynomial [25]

$$L_n^{(\alpha)}(\lambda_0 x;q)=\sum_{k=0}^{n}c_k(-\lambda_0 x)^k=$$
$$=[[\alpha+n]]_q!\sum_{k=0}^{n}(-\lambda_0 x)^k\frac{q^{k(k+\alpha)}}{[[k]]_q!}\frac{1}{[[n-k]]_q![[\alpha+k]]_q!} \tag{8}$$

as well as to q-exponential function [11, 26]

$$\exp_q\left(-\frac{\lambda_0}{2}x\right)=\sum_{n=0}^{\infty}\left(-\frac{\lambda_0 x}{2}\right)^n\frac{1}{[[n]]_q!} \tag{9}$$

where $x=r$ and $x=r^2$, respectively. In this context λ_0 is a parameter which remains to be fixed in accord with (6) and (7). However, we have to keep in mind the fact that this latter parameter gets implemented both in a system and q-dependent manner.

3. THE COULOMB PROBLEM

We shall begin by inserting the Coulomb-potential $V_1(r)=-Z/r$ and the radial wavefunction

$$f_l^{(1)}(r;q)=\exp_q\left(-\frac{\lambda_0}{2}r\right)L_n^{(2L_1)}(\lambda_0 r;q), \tag{10}$$

into (1). It is clear that (10) plays the role of a q-deformed counterpart of (6). This yields

$$H_q^{(1)}f_{l_1}^{(1)}(r;q)=E_q^{(1)}f_{l_1}^{(1)}(r;q)=\exp_q\left(-\frac{\lambda_0}{2}r\right)\tilde{H}_q^{(1)}L_n^{(2L_1)}(\lambda_0 r;q) \tag{11}$$

where now $l=l_1$, $\alpha=2L_1$ and $x=r$. One obtains

$$\tilde{H}_q^{(1)} L_n^{(2L_q)}(\lambda_0 r;q) = \frac{\mu^2}{(q+1)^2}\Big[-q^{2L_1+1}\partial_r^{(q)^2} L_n^{(2L_1)}(\lambda_0 r;q) +$$

$$+q^{2L_1+1}\frac{q+1}{2q}\partial_r^{(q)} L_n^{(2L_1)}(\lambda_0 rq;q) + [[2L_1+1]]_q \frac{\lambda_0}{2r} L_n^{(2L_1)}(\lambda_0 rq;q) - \tag{12}$$

$$-[[2L_1+1]]_q \frac{1}{r}\partial_r^{(q)} L_n^{(2L_1)}(\lambda_0 r;q) - \frac{\tilde{Z}}{r} L_n^{(2L_1)}(\lambda_0 r;q) - \frac{\lambda_0^2}{4} q^{2L_1+1} L_n^{(2L_1)}(\lambda_0 rq^2;q)\Big],$$

in which $\tilde{Z} = (q+1)^2 Z/\mu^2$. Equalizing the coefficients of the three singular $1/r$-terms gives the identification

$$\lambda_0 = \lambda_1 \equiv \frac{\tilde{Z}}{d_1^{(q)}}, \tag{13}$$

where

$$d_1^{(q)} \equiv q^{2L_1+1}[[n]]_q + \frac{1}{2}[[2L_1+1]]_q, \tag{14}$$

expresses the present q-deformation of the principal quantum-number d_1. It can be easily verified that (12) can be rewritten equivalently as

$$\tilde{H}_q^{(1)} L_n^{(2L_1)}(\lambda_0 r;q) = \frac{\lambda_0^2\mu^2}{(q+1)^2}\sum_{k=0}^{n}(-\lambda_0 r)^k c_k \cdot$$

$$\cdot\Big\{ d_1^{(q)} q^{2L_1+2k+1}\frac{s_{11}^{(q)}}{[[k+1]]_q} - q^{4L_1+4k+4} s_{12}^{(q)} s_{13}^{(q)} - \frac{1}{2} q^{2L_1+3k+1} s_{11}^{(q)} s_{13}^{(q)} - \tag{15}$$

$$-q^{6L_1+4k+5} s_{12}^{(q)} - \frac{1}{2} q^{4L_1+3k+1}(q+1) s_{11}^{(q)} - \frac{1}{4} q^{2L_1+2k+1}\Big\},$$

where

$$s_{11}^{(q)} = \frac{[[n-k]]_q}{[[2L_1+k+1]]_q}, \tag{16}$$

$$s_{12}^{(q)} = \frac{[[n-k]]_q [[n-k-1]]_q}{[[2L_1+k+1]]_q [[2L_1+k+2]]_q}, \tag{17}$$

and

$$s_{13}^{(q)} = \frac{[[2L_1+1]]_q}{[[k+1]]_q}. \tag{18}$$

Now we are ready to remark that the sum of the first five terms within the curly bracket in (15) exhibits positive values for $0 \le k \le n-1$. Such values decrease to zero as L_1 increases. This means that the last term in the curly bracket, *i.e.*, $t_6^{(k)} = -q^{2L_1+2k+1}/4$, has the meaning of a quasiclassical minimum characterizing the $\tilde{H}_{\mathrm{q}}^{(1)}$ Hamiltonian. Indeed, if $k = n$ all the five terms mentioned above become zero, so that $t_6^{(n)}$ stands for a $k = n$ realization of the quasiclassical minimum. Furthermore, using (14)–(17), we can verify that the eigenvalue-equation

$$\tilde{H}_q^{(1)} L_n^{(2L_1)}\left(\lambda_0 r; q\right) = E_q^{(1)} L_n^{(2L_1)}\left(\lambda_0 r; q\right), \tag{19}$$

can be converted into

$$\begin{gathered}
-\frac{1}{4}q^{2L_1+1}\left[qL_n^{(2L_1)}\left(\lambda_0 rq^3;q\right) - L_n^{(2L_1)}\left(\lambda_0 rq^2;q\right)\right] + \\
+q^{2L_1+1}(q-1)d_1^{(q)}L_{n-1}^{2L_1+1}\left(\lambda_0 rq^2;q\right) - \\
-\frac{1}{2}q^{2L_1+1}\left[\left[2L_1+1\right]\right]_q (q-1)L_{n-1}^{2L_1+1}\left(\lambda_0 rq^3;q\right) - \\
-q^{4L_1+4}\left[\left[2L_1+1\right]\right]_q (q-1)L_{n-2}^{2L_1-2}\left(\lambda_0 rq^4;q\right) - \\
-\frac{1}{2}q^{4L_1+1}(q+1)\left[qL_{n-1}^{2L_1+1}\left(\lambda_0 rq^4;q\right) - L_{n-1}^{2L_1+1}\left(\lambda_0 rq^3;q\right)\right] - \\
-q^{6L_1+5}\left[qL_{n-2}^{2L_1+2}\left(\lambda_0 rq^5;q\right) - L_{n-2}^{2L_1+2}\left(\lambda_0 rq^4;q\right)\right] = \\
= \frac{\tilde{E}_q^{(1)}}{\lambda_0^2}\left[qL_n^{(2L_1)}\left(\lambda_0 rq;q\right) - L_n^{(2L_1)}\left(\lambda_0 r;q\right)\right],
\end{gathered} \tag{20}$$

where $\tilde{E}_q^{(1)} = (q+1)^2 E_q^{(1)}/\mu^2$. Of course, the above results remains valid if one inserts another deformation parameter, *i.e.*, $q_1 = q^2$, instead of q.

Next we have to remark that the q-Laguerre polynomials in (20) for which the subscript is smaller than "2n" are ruled out under the limit $r \to \infty$. Combining this latter limit with (13), (19) and (20) gives

$$E_q^{(1)} = -\frac{(q+1)^2}{\mu^2} \cdot \frac{Z^2 q^{2d_1}}{\left(2d_1^{(q)}\right)^2}, \tag{21}$$

in accord with the "recipe" mentioned before [12]. Of course, resorting to the quasiclassical minimum just referred to above yields the same energy, now proceeding *via* $E_q^{(1)} = \lambda_1^2 \mu^2 t_6^{(n)} / (q+1)^2$. In other words, Eq. (21) can be interpreted quite reasonably as a quasiclassical version of the q-deformed Coulomb energy established before [1] by accounting for the (unseparated) Hamiltonian in the three-dimensional NCES. The question of whether such results can be viewed as alternative candidates for exact ones is not at all guaranteed, even if the answer is positive within the classical $q = 1$ limit. Such conjectures are well known in the usual quasiclassical WKB and $1/N$ descriptions of the Coulomb and harmonic oscillator systems [27]. One realizes, of course, that $E_q^{(1)}$ is not identical, but it compares favorably with the exact result

$$\varepsilon_q^{(1)} = -\frac{(q+1)^2}{\mu^2} \frac{Z^2 q^{2d_1}}{\left[\left[2d_1\right]\right]_q^2} \tag{22}$$

established before [1, 8]. Indeed, it can be verified that the quotient $R = 2d_1^{(q)} / \left[\left[2d_1\right]\right]_q$ approaches quite safely unity. Of course, one has $2d_1^{(q)} = [[2d_1]]_q$ if $n_r = 0$. Accordingly, higher degrees of accuracy are actually exhibited for L_1 values, which are relatively much larger than the ones of n_r. We can then say that $E_q^{(1)}$ represents a useful quasiclassical q-Laguerre alternative to $\varepsilon_q^{(1)}$.

4. THE HARMONIC OSCILLATOR

Proceeding again by virtue of the classical analogy, we shall choose the q-deformed radial wavefunction of the harmonic oscillator $\left(V_2(r) = \omega_0^2 r^2\right)$ as

$$f_{l_2}^{(2)}(x;q) = \exp_{q_1}\left(-\frac{\lambda_0}{2}x\right) L_n^{(L_2)}(\lambda_0 x; q_1), \tag{23}$$

where $l = l_2$, $x = r^2$ and $q_1 = q^2$. On the other hand one has the relationships

$$\partial_r^{(q)} = (q+1)\sqrt{x}\,\partial_x^{(q_1)}, \tag{24}$$

and

$$\partial_r^{(q)^2} = (q+1)\partial_x^{(q_1)} + q(q+1)^2 x \partial_x^{(q_1)^2}, \tag{25}$$

which are able to be used in order to handle properly the eigenvalue problem

$$\tilde{H}_q^{(2)} f_{l_2}^{(2)}(x;q_1) = E_q^{(2)} f_{l_2}^{(2)}(x;q_1). \tag{26}$$

Indeed, combining (1) with (23)–(25) and performing *a posteriori* the division by x one finds

$$\left[-q_1^{L_2+1}\partial_x^{(q_1)^2}-[[L_2+1]]_{q_1}\frac{1}{x}\partial_x^{(q_1)}-\frac{E_q^{(2)}}{\mu^2 x}\right]f_{l_2}^{(2)}(x;q_1)=-\frac{\omega_0^2}{\mu^2}f_{l_2}^{(2)}(x;q_1), \quad (27)$$

which represents the q-deformed wave equation for a Coulomb-system. Indeed, (27) is produced by (11) under the conversions $r\to x=r^2$, $2L_1\to L_2$, $2d_1\to d_2$ and $q\to q_1=q^2$, which works in conjunction with the matching conditions

$$\frac{\omega_0^2}{\mu^2}=-\tilde{E}_{q^2}^{(1)}=-E_{q^2}^{(1)}\frac{\left(q^2+1\right)^2}{\mu^2}, \quad (28)$$

and

$$\frac{E_q^{(2)}}{\mu^2}=\tilde{Z}=Z\frac{\left(q^2+1\right)^2}{\mu^2}. \quad (29)$$

In addition, we have to account for the substitution

$$2d_1^{(q)}\to d_2^{(q_1)}=2d_1^{(q^2)}=2q_1^{L_2+1}[[n]]_{q_1}+[[L_2+1]]_{q_1} \quad (30)$$

which expresses the q-deformation of the principal quantum number d_2. Then the q-deformed energy of the harmonic oscillator is given by

$$E_q^{(2)}=\frac{\mu\omega_0}{q^{d_2}}d_2^{(q_1)}, \quad (31)$$

which exhibits safely the classical limit $2\omega_0 d_2$. This result competes again with the exact eigenvalue

$$\varepsilon_q^{(2)}=\frac{\mu\omega_0}{q^{d_2}}[[d_2]]_{q^2} \quad (32)$$

established elsewhere before [4, 5] on the N-dimensional NCES.

However, there is still an open point, namely the specification of the λ_0 parameter in (23). This amounts to establish the oscillator counterpart of (13). Starting from the factorization

$$H_q^{(2)}f_{l_2}^{(2)}(x;q_1)=\exp_{q_1}\left(-\frac{\lambda_0}{2}x\right)\tilde{H}_{q_1}^{(2)}L_n^{(L_2)}(\lambda_0 x;q_1), \quad (33)$$

and proceeding as in Section 3, one finds

$$\begin{gathered}
-xq_1^{L_2+1}\partial_x^{(q_1)^2} L_n^{(L_2)}\left(\lambda_0 x;q_1\right)+ \\
+\lambda_0\frac{q_1+1}{2q_1}q_1^{L_2+1}x\partial_x^{(q_1)}L_n^{(L_2)}\left(\lambda_0 x;q_1\right)-\frac{\lambda_0^2}{4}q_1^{L_2+1}xL_n^{(L_2)}\left(\lambda_0 xq_1^2;q_1\right)- \\
-\left[\left[L_2+1\right]\right]_{q_1}\partial_x^{(q_1)}L_n^{(L_2)}\left(\lambda_0 x;q_1\right)+\frac{\lambda_0}{2}\left[\left[L_2+1\right]\right]_{q_1}L_n^{(L_2)}\left(\lambda_0 xq_1;q_1\right)+ \\
+\frac{\omega_0^2}{\mu^2}xL_n^{(L_2)}\left(\lambda_0 x;q_1\right)=\frac{E_q^{(2)}}{\mu^2}L_n^{(L_2)}\left(\lambda_0 x;q_1\right).
\end{gathered} \tag{34}$$

One sees that there are two x^{n+1}-contributions such as produced by the third and sixth terms in the l.h.s. of (34). Equalizing the coefficients yields

$$\lambda_0=\lambda_2\equiv\frac{2\omega_0}{\mu q^{d_2}}, \tag{35}$$

so that (23) is completely specified. We have to remark that (35) is also produced by combining the q^2-version of (13) with (28). Moreover, we can resort once more again to the quasiclassical $r = 0$ limit, which reproduces precisely Eq. (31), in this way also. Such agreements are able to support the relevance of the present calculations.

5. CONCLUSIONS

In this paper the q-deformed radial equations for the Coulomb and harmonic oscillator systems have been treated in terms of products between q-Laguerre polynomials and q-exponential functions. Such q-Laguerre polynomials can be normalized by resorting to weighted or q-integrals, respectively [11, 22]. Related q-deformed spherical harmonics have been already discussed before [1, 2, 7, 9], but a more general description in terms of q-Gegenbauer polynomials [28] is of further interest. Unlike some immediate expectations, the q-deformed energies established in a quasiclassical manner in this paper are not identical to the exact ones derived before with the help of operator methods acting on the NCES. Nevertheless, such energies are able to exhibit high degrees of accuracy. Moreover, the present results are exact ones for the ground-state ($n_r = 0$), which sheds some light on the capabilities of the present q-deformed radial description. The present q-deformed principal quantum numbers get represented specifically under modified forms, as indicated by (14) and (30) respectively. In order to interpolate between descriptions based on (21) and (22), the derivation of inter-connection curves in the (q, q') plane for which $2d_1^{(q')}=[[2d_1]]_q$ deserves further attention. Obviously, a similar curve can be derived for the harmonic oscillator. We could then say that more refined versions of q-deformed radial wavefunctions are in order if $n_r\neq 0$, but such issues remain open for further discussions. So far we have learned that proceeding by virtue of the classical analogy does not ensure the onset of the exact results for $n_r \neq 0$, as one might expect on general reasons concerning unexpected peculiarities of q-deformations.

The main point is, however, the fact that by now q-deformed energy results such as done by (21) and (31) work in conjunction with explicit q deformed wavefunctions, *i.e.*, in connection with (10) and (23), respectively. We emphasize that such explicit results are ready to be used for further applications, like the ones referring to the quantum Hall effect. In this latter context generalizations towards q deformation parameters which are roots of unity deserve further attention.

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REFERENCES

1. X.-C. Song and L. Liao, J. Phys., A **25**, 623 (1992).
2. J. Feigenbaum and P. G. O. Freund, J. Math. Phys., **37**, 1602 (1996).
3. G. Fiore, Int. J. Mod. Phys., A **8**, 4679 (1993).
4. U. C. Watamura and S. Watamura, Int. J. Mod. Phys., A **9**, 3989 (1994).
5. U. C. Watamura and S. Watamura, Prog. Theor. Phys. Suppl., **118**, 375 (1995).
6. Z. Z. Zhong, Phys. Rev., A **55**, 3341 (1997).
7. M. Micu, J. Phys., A **32**, 7765 (1999).
8. E. Papp, J. Phys., A **29**, 1795 (1996).
9. E. Papp, *Derivation of q Analogs for the Radial Schrödinger Equation in N Space Dimensions*, Nova Science, New-York (1997).
10. E. Papp, Phys. Rev., A **52**, 101 (1995).
11. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge (1990).
12. G. Chan, R. Finkelstein and V. Oganesyan, J. Math. Phys., **38**, 2132 (1997).
13. J. Bellissard, A. van Elst and H. Schulz-Baldes, J. Math. Phys., **35**, 5373 (1994).
14. Ö. F. Dayi and A. Jellal, Phys. Lett., A **287**, 349 (2001).
15. A. Jellal, J. Phys., A **34**, 10159 (2001).
16. E. Papp, J. Phys., A **35**, 3337 (2002).
17. A. Lavagno and P. N. Swamy, Phys. Rev., **E 64**, 036101 (2002).
18. M. Lorente, Phys. Lett., A **285**, 119 (2001).
19. A. Turbiner, Int. J. Mod. Phys., A **16**, 1579 (2001).
20. M. Irac-Astaud and G.Rideau, Rev. Math. Phys., **11**, 1061 (1999).
21. R. J. Finkelstein, Int. J. Mod. Phys., A **13**, 1795 (1998).
22. J. Wess, Int. J. Mod. Phys. A **13**, 4987 (1998).
23. V. A. Kostelecky and N. Russell, J. Math. Phys., **37**, 2166 (1996); V. A. Kostelecky, M. M. Nieto and D. R. Truax, Phys. Rev., D **32**, 2627 (1985).
24. E. Papp, Phys. Rev., A **38**, 5910 (1988).
25. D. S. Moak, J. Math. Anal. Appl., **81**, 20 (1981).
26. R. Floreanini and L. Vinet, J. Math. Phys., **36**, 3134 (1995).
27. J. B. Krieger and C. Rosenzweig, Phys. Rev., **164**, 171 (1967), T. Imbo and U. Sukhatme, Phys. Rev., D **31**, 2655 (1985) and E. Papp, Phys. Rev., A **38**, 2158 (1988).
28. R. Floreanini and L. Vinet, Contemporary Mathematics, **160**, 85 (1994).