

An upper bound on the maximal probability for successful conversion of multipartite pure entangled states

Iulia Ghiu,^{*1} Mohamed Bourennane,² and Anders Karlsson²

¹ Department of Physics, University of Bucharest,

P.O. Box MG-11, R-077125, Bucharest-Măgurele, Romania

²Department of Microelectronics and Information Technology,

Royal Institute of Technology (KTH),

Electrum 229, 164 40 Kista, Sweden

Received June 22, 2005

Abstract

We present a proposition that gives an upper bound on the maximal probability for conversion of multipartite pure entangled states using only local operations and classical communication. Then, we apply the proposition to some examples and show that it is possible to find the optimal transformation for particular cases.

Key words: entangled states, local transformations, Nielsen theorem

1 Introduction

The study of multipartite entanglement has recently become of increasing interest in quantum information theory. One of the main goals is to characterize quantitatively and qualitatively multipartite entanglement. While for two-particle states, one copy entanglement manipulation is known, see e.g.

*Email: iughiu@barutu.fizica.unibuc.ro

Refs.[1, 2], for multiparticle entanglement there still remain open questions concerning both classification as well as optimal local manipulations.

Nielsen has found the necessary and sufficient condition for conversion of bipartite pure state, when only local operations and classical communication (LOCC) are allowed [2]. A more general treatment of local manipulations of bipartite pure states was subsequently given by Vidal [3]. It has been shown that there are two classes of pure tripartite entangled states: the GHZ-class (states of the form $|GHZ\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)$) and the W-class (states of the form $1/\sqrt{3}(|100\rangle + |010\rangle + |001\rangle)$), which are non-interchangeable under stochastic LOCC [4]. Dür has investigated the entanglement properties of N-partite states when tracing out any $N - 2$ particles and he has shown that there exist states where all particles are entangled with all particles [5]. In the last years, interesting problems related to entanglement-assisted transformations and multiple-copy entanglement transformations have been analyzed [6, 7, 8].

In this work we study the optimal conversion of a multipartite state into another one. This paper is organized as follows: We start by reviewing the local manipulation protocols of bipartite entangled states, and also the concept of bipartite splitting of N parties [9]. Then, using the existence of a Schmidt decomposition of a bipartite system [10], we find an upper bound on the maximal probability for conversion between the two states.

Since for the case when the number of particles $N \geq 3$, we know only the optimal manipulation between two tripartite states of the GHZ-type, namely the optimal distillation of a GHZ-state [11], we apply our result on an example studied in Ref. [11], and show that we recover the same maximal probability using a simpler method. Then, we prove that for particular situations we can compute the maximal probability of success for conversion between two states by employing the proposition.

2 Local manipulation of bipartite entangled states

Let us start reviewing Nielsen's theorem given in Ref.[2]. Consider two pure bipartite states $|\psi^1\rangle$ and $|\psi^2\rangle$ shared by the usual parties, Alice and Bob.

The Schmidt decompositions of the two states are:

$$|\psi^{(m)}\rangle = \sum_{i=1}^n \sqrt{\alpha_i^{(m)}} |i_A^{(m)} i_B^{(m)}\rangle, \tag{2.1}$$

where $\alpha_i^{(m)} \geq \alpha_{i+1}^{(m)} \geq 0$, $\sum_{i=1}^n \alpha_i^{(m)} = 1$, and $m = 1, 2$. Without loss of generality, the state spaces of the two particles are supposed to have the same dimension, n .

It has been shown that a necessary and sufficient condition for the local conversion with probability one of the state $|\psi^{(1)}\rangle$ into the state $|\psi^{(2)}\rangle$ is that the vector $\vec{\alpha}^{(1)} = (\alpha_1^{(1)}, \dots, \alpha_n^{(1)})$ is majorized by the vector $\vec{\alpha}^{(2)} = (\alpha_1^{(2)}, \dots, \alpha_n^{(2)})$:

$$\sum_{i=1}^k \alpha_i^{(1)} \leq \sum_{i=1}^k \alpha_i^{(2)}, \quad k = 1, \dots, n. \tag{2.2}$$

Vidal has shown that the maximal probability for conversion of $|\psi^{(1)}\rangle$ into $|\psi^{(2)}\rangle$, $P(|\psi^{(1)}\rangle \rightarrow |\psi^{(2)}\rangle)$ by LOCC is [3]:

$$P(|\psi^{(1)}\rangle \rightarrow |\psi^{(2)}\rangle) = \min_{l \in [1, n]} \frac{E_l(\psi^{(1)})}{E_l(\psi^{(2)})}, \tag{2.3}$$

where E_l are entanglement monotones defined by: $E_l(|\psi^{(i)}\rangle) = \sum_{k=l}^n \alpha_k^{(i)}$, which are quantities which do not increase under LOCC, on average.

3 Multipartite entangled states

Let us now introduce the concept of bipartite splitting: consider N observers who share N particles. A partition of the N particles into two groups is called a bipartite splitting if the observers who have the particles in the same group can act jointly on them.

Proposition:

Let $|\Phi\rangle$ and $|\Psi\rangle$ be two N -particle pure states shared by N observers $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_N$. The state space is $\mathcal{H} = \mathcal{C}^{d_1} \otimes \mathcal{C}^{d_2} \otimes \dots \otimes \mathcal{C}^{d_N}$. We denote by $|\chi\rangle_{i_1, i_2, \dots, i_m}$ a m -particle state which belongs to the observers $\mathcal{A}_{i_1}, \dots, \mathcal{A}_{i_m}$, who can act jointly on this state. Then the maximal probability $P(|\Phi\rangle \rightarrow |\Psi\rangle)$ to convert $|\Phi\rangle$ into $|\Psi\rangle$ by local operations and classical communication (LOCC) satisfies :

$$P(|\Phi\rangle \rightarrow |\Psi\rangle) \leq \min\{P(|\Phi_{\Pi(1), \dots, \Pi(m)}^{bipartite}\rangle \rightarrow |\Psi_{\Pi(1), \dots, \Pi(m)}^{bipartite}\rangle)\}, \tag{3.4}$$

where we have denoted by $|\chi_{\Pi(1),\dots,\Pi(m)}^{bipartite}\rangle$ the state which describes an arbitrary bipartite splitting of the initial and final states

$$|\chi_{\Pi(1),\dots,\Pi(m)}^{bipartite}\rangle = \sum_k \sqrt{p_k} |\chi_k\rangle_{\Pi(1),\dots,\Pi(m)} |\chi_k\rangle_{\Pi(m+1),\dots,\Pi(N)}, \tag{3.5}$$

$|\chi\rangle = |\Psi\rangle, |\Phi\rangle$. The minimization in Eq.(3.4) is to be performed over $m = 1, 2, \dots, N$ and all the permutations Π of N particles, and $P(|\Phi_{\Pi(1),\dots,\Pi(m)}^{bipartite}\rangle \rightarrow |\Psi_{\Pi(1),\dots,\Pi(m)}^{bipartite}\rangle)$ is the maximal probability (2.3) for conversion of the bipartite states.

Proof. Consider the minimal probability in Eq.(3.4) is obtained when $m = n$ and $\Pi = \begin{pmatrix} 1 & 2 & \dots & N \\ i_1 & i_2 & \dots & i_N \end{pmatrix}$. We will denote by $\mathcal{A}_{i_1,\dots,i_n}$ the observer who acts on the state $|\chi\rangle_{i_1,i_2,\dots,i_n}$, and by $\mathcal{A}_{i_{n+1},\dots,i_N}$ another observer who acts on the state $|\chi\rangle_{i_{n+1},i_{n+2},\dots,i_N}$.

Suppose that there is a protocol \mathcal{P} for conversion $|\Phi\rangle$ into $|\Psi\rangle$ by LOCC with a probability of success $P(|\Phi\rangle \rightarrow |\Psi\rangle)$ such that:

$$P(|\Phi\rangle \rightarrow |\Psi\rangle) > P(|\Psi_{i_1,\dots,i_n}^{bipartite}\rangle \rightarrow |\Phi_{i_1,\dots,i_n}^{bipartite}\rangle). \tag{3.6}$$

Then the two observers $\mathcal{A}_{i_1,\dots,i_n}$ and $\mathcal{A}_{i_{n+1},\dots,i_N}$ can use the protocol \mathcal{P} in order to perform the transformation, which would give a greater probability than $P(|\Psi_{i_1,\dots,i_n}^{bipartite}\rangle \rightarrow |\Phi_{i_1,\dots,i_n}^{bipartite}\rangle)$. However the probability $P(|\Psi_{i_1,\dots,i_n}^{bipartite}\rangle \rightarrow |\Phi_{i_1,\dots,i_n}^{bipartite}\rangle)$, obtained using the Eq.(2.3), is the maximal probability for conversion of the two bipartite states and this leads to contradiction.

Corollary

If we can find a local protocol that gives a probability of success for conversion between two states equal to the upper bound given by Eq.(3.4), then this protocol is optimal.

The corollary gives the usefulness of the proposition. It allows us to test if a certain protocol we have is optimal or not. At the same time, however, the proposition has a major weakness. It does not say whether LOCC can reach the upper bound or not, and in a bad case it may be that generally the upper bound given by the proposition is too high, then by making the proposition less useful. Specific examples are given in the next section, both giving useful results, as well as results where a local protocol does not reach the bound.

Let us compute the minimum number of bipartite splittings [9] we have

to consider in order to apply the proposition:

$$n = \frac{1}{2} \sum_{m=1}^N \binom{N}{m} = 2^{N-1} - 1. \quad (3.7)$$

4 Local transformations between three-particle states

In the following we will apply the above proposition to two examples of three-particle entangled states shared by three persons, Alice, Bob, and Charlie, hence we have to compute the probabilities for the three cases of bipartite splitting: (i) 'Alice and Bob' – 'Charlie'; (ii) 'Bob and Charlie' – 'Alice'; (iii) 'Alice and Charlie' – 'Bob'.

Example 1. Consider that three observers, Alice, Bob, and Charlie, share a W-state:

$$|\psi_W\rangle = \sqrt{a}|100\rangle + \sqrt{b}|010\rangle + \sqrt{c}|001\rangle, \quad (4.8)$$

where $a, b, c > 0$, $a+b+c = 1$, and $a \geq b \geq c$. They want to obtain a bipartite maximally entangled state shared by Alice and Bob: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB}|0\rangle_C$.

(i) The Schmidt decomposition of the initial state (4.8) is:

$$|\psi_W\rangle = \sqrt{a+b} \left(\sqrt{\frac{a}{a+b}}|10\rangle + \sqrt{\frac{b}{a+b}}|01\rangle \right)_{AB} |0\rangle_C + \sqrt{c}|00\rangle_{AB}|1\rangle_C. \quad (4.9)$$

The maximal probability for conversion of the bipartite state (4.9) into a EPR pair is given by Eq.(2.3): $P_1 = 1$.

(ii) The Schmidt decomposition is in this case:

$$|\psi_W\rangle = \sqrt{a}|1\rangle_A|00\rangle_{BC} + \sqrt{b+c}|0\rangle_A \left(\sqrt{\frac{b}{b+c}}|10\rangle + \sqrt{\frac{c}{b+c}}|01\rangle \right)_{BC}. \quad (4.10)$$

The maximal probability is: $P_2 = 2\min\{a, 1-a\}$.

(iii) The state (4.8) can be written in its Schmidt decomposition:

$$|\psi_W\rangle = \sqrt{a+c} \left(\sqrt{\frac{a}{a+c}}|10\rangle + \sqrt{\frac{c}{a+c}}|01\rangle \right)_{AC} |0\rangle_B + \sqrt{b}|00\rangle_{AC}|1\rangle_B. \quad (4.11)$$

Then, we have: $P_3 = 2b$. The above proposition gives the upper bound: $P(|\psi_W\rangle \rightarrow |EPR\rangle_{AB}) \leq 2b$.

We will show that there is a local protocol that gives a probability equal to the upper bound corresponding to the case a). First, Charlie measures his particle in the computational basis, and will obtain the outcome $|0\rangle$ with a probability equal to $'a + b'$, therefore the total state will be projected onto:

$$\left(\sqrt{\frac{a}{a+b}}|10\rangle + \sqrt{\frac{b}{a+b}}|01\rangle \right)_{AB} |0\rangle_C. \quad (4.12)$$

He has to communicate the outcome of his measurement (one classical bit of information) to Alice and Bob. Second, Alice and Bob will convert locally the state (4.12) into an EPR pair with a maximal probability [3] equal to $'\frac{2b}{a+b}'$. The total probability for obtaining one EPR pair is equal to $'2b'$, and according to the corollary, the above local protocol is optimal.

The next example has been studied in Ref.[11], where it has been presented the optimal distillation of the GHZ-state from one copy of an arbitrary state which belongs to the GHZ-class.

Example 2. Consider that three observers, Alice, Bob, and Charlie, share the three-particle state:

$$|\phi^1\rangle = \mu_1|00c_1\rangle + \mu_2|11c_2\rangle, \quad \mu_1 \geq \mu_2 > 0, \quad (4.13)$$

where the particles "1", "2", and "3" belong to Alice, Bob, and Charlie, respectively. They want to transform this state into the GHZ-state $|GHZ_{(3)}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. We can write the state of Charlie's particle as following:

$$|c_i\rangle = \gamma_i|0\rangle + \delta_i|1\rangle, \quad \gamma_i, \delta_i \geq 0, \quad \gamma_i^2 + \delta_i^2 = 1, \quad i = 1, 2. \quad (4.14)$$

(i) In this case, the state space is $\mathcal{C}^4 \otimes \mathcal{C}^2$, where \mathcal{C}^4 represents the space of particles "1" and "2", and can be described by the basis:

$$|00\rangle = |\bar{0}\rangle, |01\rangle = |\bar{1}\rangle, |10\rangle = |\bar{2}\rangle, |11\rangle = |\bar{3}\rangle. \quad (4.15)$$

Therefore, the state (4.13) can be written:

$$|\phi^1\rangle = \mu_1\gamma_1|\bar{00}\rangle + \mu_1\delta_1|\bar{01}\rangle + \mu_2\gamma_2|\bar{30}\rangle + \mu_2\delta_2|\bar{31}\rangle, \quad (4.16)$$

and the GHZ-state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|\bar{0}0\rangle + |\bar{3}1\rangle). \quad (4.17)$$

We will denote by 'y' the following scalar product:

$$\langle c_1 | c_2 \rangle = y. \quad (4.18)$$

The maximal probability is given by Eq.(2.3):

$$P_1 = 1 - \sqrt{1 - 4\mu_1^2\mu_2^2(1 - y^2)}. \quad (4.19)$$

(ii) The state space is $\mathcal{C}^2 \otimes \mathcal{C}^4$, where \mathcal{C}^4 represents the space of particles "2" and "3", and can be described by the basis (4.15). The Schmidt decomposition of $|\phi^1\rangle$ is:

$$|\phi^1\rangle = \mu_1|0\rangle(\gamma_1|\bar{0}\rangle + \delta_1|\bar{1}\rangle) + \mu_2|1\rangle(\gamma_2|\bar{2}\rangle + \delta_2|\bar{3}\rangle). \quad (4.20)$$

The maximal probability is: $P_2 = 2\mu_2^2$.

(iii) The state space is $\mathcal{C}^4 \otimes \mathcal{C}^2$, where \mathcal{C}^4 represents the space of particles "1" and "3", and can be described by the basis (4.15). The Schmidt decomposition of $|\phi^1\rangle$ is:

$$|\phi^1\rangle = \mu_1(\gamma_1|\bar{0}\rangle + \delta_1|\bar{1}\rangle)|0\rangle + \mu_2(\gamma_2|\bar{2}\rangle + \delta_2|\bar{3}\rangle)|1\rangle. \quad (4.21)$$

The maximal probability is: $P_3 = 2\mu_2^2$. The minimal probability $\min\{P_1, P_2, P_3\}$ is obtained in the first case and is equal to P_1 given by Eq.(4.19).

Let us consider the following transformation:

$$I \otimes I \otimes (\epsilon_1|0\rangle\langle c'_1| + \epsilon_2|1\rangle\langle c'_2|), \quad (4.22)$$

where $\{|c'_1\rangle, |c'_2\rangle\}$ is the biorthonormal basis to $|c_1\rangle, |c_2\rangle$, and

$$\epsilon_i = \frac{1}{\sqrt{2}\mu_i} \left(1 - \sqrt{1 - 4\mu_1^2\mu_2^2(1 - y^2)}\right)^{1/2}, i = 1, 2. \quad (4.23)$$

By applying this local operator onto the state $|\phi^1\rangle$, the three observers will get the GHZ-state with a probability equal to the upper bound P_1 given by Eq.(4.19), hence this protocol is optimal.

5 Transformations between N-particle states

Example 3. Let us now suppose that Alice, Bob, and Charlie share a copy of the following W-state (the three particles are denoted by the indices "1", "2", and "3"):

$$|\psi_W\rangle_{123} = (\sqrt{a}|100\rangle + \sqrt{a}|010\rangle + \sqrt{1-2a}|001\rangle)_{123}, \quad (5.24)$$

where $a \in [\frac{1}{3}, \frac{1}{2})$, and, at the same time, Bob and Charlie have an EPR pair, the total state being: $|\phi^2\rangle = |\psi_W\rangle_{123} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{45}$. They want to perform the transformation between the two inequivalent classes of three-particle entangled states with the help of one EPR pair [12]: $|\phi^2\rangle \rightarrow |\chi\rangle$, where

$$|\chi\rangle = |GHZ\rangle_{125} \otimes |00\rangle_{34} = \frac{1}{\sqrt{2}}|0\rangle_1|00\rangle_{24}|00\rangle_{35} + \frac{1}{\sqrt{2}}|1\rangle_1|10\rangle_{24}|01\rangle_{35}. \quad (5.25)$$

Since we have three observers, we have to consider again the three possibilities of bipartite splitting as before.

Case (i): The initial state can be written in the Schmidt decomposition:

$$|\phi^2\rangle = \sqrt{a}|\bar{0}\rangle_{AB}|\bar{0}\rangle_C + \sqrt{a}|\bar{1}\rangle_{AB}|\bar{1}\rangle_C + \sqrt{\frac{1-2a}{2}}|\bar{2}\rangle_{AB}|\bar{2}\rangle_C + \sqrt{\frac{1-2a}{2}}|\bar{3}\rangle_{AB}|\bar{3}\rangle_C, \quad (5.26)$$

where $|\bar{0}\rangle_{AB} = \frac{1}{\sqrt{2}}(|1\rangle|00\rangle + |0\rangle|10\rangle)$, $|\bar{1}\rangle_{AB} = \frac{1}{\sqrt{2}}(|1\rangle|01\rangle + |0\rangle|11\rangle)$, $|\bar{2}\rangle_{AB} = |0\rangle|00\rangle$, $|\bar{3}\rangle_{AB} = |0\rangle|01\rangle$, $|\bar{0}\rangle_C = |00\rangle$, $|\bar{1}\rangle_C = |01\rangle$, $|\bar{2}\rangle_C = |10\rangle$, $|\bar{3}\rangle_C = |11\rangle$.

The Schmidt decomposition of the final state is:

$$|\chi\rangle = \frac{1}{\sqrt{2}}|0'\rangle_{AB}|\bar{0}\rangle_C + \frac{1}{\sqrt{2}}|1'\rangle_{AB}|\bar{1}\rangle_C, \quad (5.27)$$

with $|0'\rangle_{AB} = |0\rangle|00\rangle$ and $|1'\rangle_{AB} = |1\rangle|10\rangle$. We will denote by $\psi_1 \sim \psi_2$ two states ψ_1, ψ_2 which are local unitary equivalent. We have:

$$|\chi\rangle \sim |\epsilon\rangle = \frac{1}{\sqrt{2}}|\bar{0}\rangle_{AB}|\bar{0}\rangle_C + \frac{1}{\sqrt{2}}|\bar{1}\rangle_{AB}|\bar{1}\rangle_C. \quad (5.28)$$

The maximal probability of success to transform $|\phi^2\rangle \rightarrow |\epsilon\rangle$ is given by Eq.(2.3): $P_1 = 1$.

Case (ii): The Schmidt decomposition is:

$$|\phi^2\rangle = \sqrt{1-a}|\bar{0}\rangle_A|\bar{0}\rangle_{BC} + \sqrt{a}|\bar{1}\rangle_A|\bar{1}\rangle_{BC}, \quad (5.29)$$

where $|\bar{0}\rangle_A = |0\rangle$, $|\bar{1}\rangle_A = |1\rangle$, $|\bar{0}\rangle_{BC} = \frac{1}{\sqrt{1-a}}(\sqrt{\frac{a}{2}}|10\rangle|00\rangle + \sqrt{\frac{a}{2}}|11\rangle|01\rangle + \sqrt{\frac{1-2a}{2}}|00\rangle|10\rangle + \sqrt{\frac{1-2a}{2}}|01\rangle|11\rangle)$, $|\bar{1}\rangle_{BC} = \frac{1}{\sqrt{2}}(|00\rangle|00\rangle + |01\rangle|01\rangle)$.

The final state is:

$$\begin{aligned} |\chi\rangle &= \frac{1}{\sqrt{2}}|\bar{0}\rangle_A|0'\rangle_{BC} + \frac{1}{\sqrt{2}}|\bar{1}\rangle_A|1'\rangle_{BC} \\ &\sim |\mu\rangle = \frac{1}{\sqrt{2}}|\bar{0}\rangle_A|\bar{0}\rangle_{BC} + \frac{1}{\sqrt{2}}|\bar{1}\rangle_A|\bar{1}\rangle_{BC}. \end{aligned} \quad (5.30)$$

The maximal probability to transform $|\phi\rangle$ into $|\mu\rangle$ is given by Eq.(2.3): $P_2 = 2a$.

Case (iii) The Schmidt decomposition of the initial state is:

$$\begin{aligned} |\phi^3\rangle &= \sqrt{\frac{1-a}{2}}|\bar{0}\rangle_{AC}|\bar{0}\rangle_B + \sqrt{\frac{1-a}{2}}|\bar{1}\rangle_{AC}|\bar{1}\rangle_B \\ &+ \sqrt{\frac{a}{2}}|\bar{2}\rangle_{AC}|\bar{2}\rangle_B + \sqrt{\frac{a}{2}}|\bar{3}\rangle_{AC}|\bar{3}\rangle_B, \end{aligned} \quad (5.31)$$

where $|\bar{0}\rangle_{AC} = \sqrt{\frac{a}{1-a}}|1\rangle|00\rangle + \sqrt{\frac{1-2a}{1-a}}|0\rangle|10\rangle$, $|\bar{1}\rangle_{AC} = \sqrt{\frac{a}{1-a}}|1\rangle|01\rangle + \sqrt{\frac{1-2a}{1-a}}|0\rangle|11\rangle$, $|\bar{2}\rangle_{AC} = |0\rangle|00\rangle$, $|\bar{3}\rangle_{AC} = |0\rangle|01\rangle$, $|\bar{0}\rangle_B = |00\rangle$, $|\bar{1}\rangle_B = |01\rangle$, $|\bar{2}\rangle_B = |10\rangle$, $|\bar{3}\rangle_B = |11\rangle$. We have in this case $P_3 = 1$.

Therefore the upper bound is equal to $P_2 = 2a$. Using the proposition we obtain the following relation for the maximal probability of conversion between the two states by LOCC:

$$P(|\phi^2\rangle \rightarrow |GHZ\rangle_{125} \otimes |00\rangle_{34}) \leq 2a. \quad (5.32)$$

In the following we will find a local protocol for conversion between the initial state and the final one, which succeeds with a probability equal to the upper bound. This protocol consists of three steps:

(a) Charlie measures his particle "3" in the computational basis. Charlie will obtain the state $|0\rangle_3$ with the probability '2a', therefore the state of the whole system will be projected onto:

$$\begin{aligned} |\phi'\rangle &= \frac{1}{2}(|1\rangle_1|00\rangle_{24}|0\rangle_5 + |1\rangle_1|01\rangle_{24}|1\rangle_5) + \\ &+ \frac{1}{2}(|0\rangle_1|10\rangle_{24}|0\rangle_5 + |0\rangle_1|11\rangle_{24}|1\rangle_5). \end{aligned} \quad (5.33)$$

(b) Bob applies a CNOT operation onto his particles, where the particle “2” represents the source, while particle “4” is the target:

$$\begin{aligned} |\phi''\rangle &= \frac{1}{2}(|1\rangle_1|0\rangle_2|0\rangle_5 + |0\rangle_1|1\rangle_2|1\rangle_5)|0\rangle_4 + \\ &+ \frac{1}{2}(|1\rangle_1|0\rangle_2|1\rangle_5 + |0\rangle_1|1\rangle_2|0\rangle_5)|1\rangle_4. \end{aligned} \quad (5.34)$$

(c) Bob measures particle “4” in the computational basis. Regardless of Bob’s outcome measurement, the final state is local unitary equivalent to the state GHZ. We have obtained the final state with the total probability: $P = 2a$. The probability given by the above local protocol is equal to the upper bound, and according to the corollary, the local protocol is optimal.

Example 4. Let us now discuss a generalization of the second example: Consider that N observers share the following state:

$$|\phi^3\rangle = \mu_1|00\dots 0c_1\rangle + \mu_2|11\dots 1c_2\rangle, \quad \mu_1 \geq \mu_2 > 0, \quad (5.35)$$

where the general states $|c_i\rangle$, $i = 1, 2$, are given by Eq.(4.14). They want to transform it into the N -particle GHZ-state: $|GHZ_{(N)}\rangle = \frac{1}{\sqrt{2}}(|0\dots 00\rangle + |1\dots 11\rangle)$.

Cases (i_k) , $k = 1, 2, \dots, N - 2$: The Schmidt decomposition of the state is:

$$\begin{aligned} |\phi_3\rangle &= \mu_1|\underbrace{0\dots 0}_k\rangle(\gamma_1|\underbrace{0\dots 00}_{N-k}\rangle + \delta_1|\underbrace{0\dots 01}_{N-k}\rangle) + \\ &+ \mu_2|\underbrace{1\dots 1}_k\rangle(\gamma_2|\underbrace{1\dots 10}_{N-k}\rangle + \delta_2|\underbrace{1\dots 11}_{N-k}\rangle). \end{aligned} \quad (5.36)$$

The maximal probability is: $P_1 = 2\mu_2^2$.

Case (i_{N-1}) : The state is written as follows:

$$\begin{aligned} |\phi^3\rangle &= \mu_1|00\dots 0\rangle|c_1\rangle + \mu_2|11\dots 1\rangle|c_2\rangle \\ &= \mu_1\gamma_1|00\dots 0\rangle|0\rangle + \mu_1\delta_1|00\dots 0\rangle|1\rangle \\ &+ \mu_2\gamma_2|11\dots 1\rangle|0\rangle + \mu_2\delta_2|11\dots 1\rangle|1\rangle. \end{aligned} \quad (5.37)$$

In this case, the maximal probability is given by:

$$P_2 = 1 - \sqrt{1 - 4\mu_1^2\mu_2^2(1 - y^2)}, \quad (5.38)$$

where ‘ y ’ is the scalar product (4.18). According to the proposition we have:

$$P(|\phi^3\rangle \rightarrow |GHZ_{(N)}\rangle) \leq 1 - \sqrt{1 - 4\mu_1^2\mu_2^2(1 - y^2)}. \quad (5.39)$$

We can generalize the operator (4.22) to the case of N-particle state as follows:

$$I \otimes I \otimes \dots I \otimes (\epsilon_1|0\rangle\langle c'_1| + \epsilon_2|1\rangle\langle c'_2|), \quad (5.40)$$

where $\{|c'_1\rangle, |c'_2\rangle\}$ is the biorthonormal basis to $|c_1\rangle, |c_2\rangle$, and $\epsilon_i, i = 1, 2$ have been defined by Eq.(4.23). The local operator (5.40) transforms the initial state into the N-particle GHZ-state with a probability equal to the upper bound, P_2 , hence this represents an optimal local protocol.

6 Conclusions

In conclusion, in this paper we have studied the one copy manipulation of multipartite pure entangled states. We have found an upper bound on the maximal probability for conversion of multipartite states. This is given by the optimal local manipulations of the bipartite splittings of the multipartite states.

We have analyzed examples of multipartite entanglement conversion and shown that it is possible to perform local transformations between two multipartite states with a probability equal to the upper bound and have concluded that these transformations represent the optimal protocol.

Acknowledgment. This work was supported by the Romanian CNCSIS through a grant for the University of Bucharest and by the Swedish foundation for strategic research-SSF.

References

- [1] D. Jonathan and M. Plenio, Phys. Rev. Lett. **83**, 3566 (1999).
- [2] M. A. Nielsen, Phys. Rev. Lett. **83**, 436, (1999).
- [3] G. Vidal, Phys. Rev. Lett. **83**, 1046 (1999).
- [4] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
- [5] W. Dür, Phys. Rev. A **63**, 020303 (2001).
- [6] P. Hayden, Phys. Rev. A **67**, 012326 (2003).
- [7] R. Duan, Y. Feng, X. Li, and M. Ying, Phys. Rev. A **71**, 042319 (2005).

- [8] S. Ishizaka and M.B. Plenio, Phys. Rev. A **71**, 052302 (2005).
- [9] W. Dür and J. I. Cirac, Phys. Rev. A **61**, 042314 (2000).
- [10] A. Ekert and P. L. Knight, Am. J. Phys. **63**, 415 (1993).
- [11] A. Acín, E. Jané, W. Dür, and G. Vidal, Phys. Rev. Lett. **85**, 4811 (2000).
- [12] I. Ghiu, M. Bourennane, and A. Karlsson, Phys. Lett. A **287**, 12 (2001).