

KINETIC SPECTRA IN A SPHERICAL CAVITY

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Abstract. Discrete values of the square of momentum operator are obtained by asking the vanishing of the wave function on the inner surface of a sphere of radius R . Discrete values of the kinetic energy of a free particle confined inside the sphere are obtained by resorting to the momentum-energy relativistic relationship. The principal features of the energy spectrum are listed and interpreted in terms of a multiplet structure somewhat analogous to the Hydrogen fine structure. The average value of the momentum over the range of the multiplet turns out to be equal with the value calculated with the asymptotic wave functions, ($P = \hbar\bar{z}/R$, $\bar{z} \approx (\pi/2)N$). The extension of multiplet range, Δz , approaches the constant value $\Delta z = \pi$ for $N \rightarrow \infty$. A statistical study, undertaken on a lot of 400 discrete values of P^2 leads to relevant results for: (a) the momentum: $P \approx hN/4R$ and (b) the multiplet weight: $g(N) \approx (\pi^2/12)(N^2 - 0.4842106N + 0.1419483)$. At the same time, adopting the ortho-normalized set of asymptotic wave functions, for classifying the discrete states in terms of the two relevant quantum numbers (N, l), we derive the following main spectral properties: α) there are some states with $l > N$, i.e., with negative radial quantum numbers: $n = (N - l)/2$; β) there are some asymptotic (N, l) states, which are missing among the components of the multiplet (called “locked states” or antiresonances). We have used these results for correct calculation of the statistical distribution of quanta or particle in the spherical cavities, in physical conditions implying a small number N of multiplets.

Key words: spherical cavities, kinetics, energy spectra, free particles.

1. INTRODUCTION

The ideal classical cavity may be defined as a closed surface with a perfectly smooth and unitary reflection interior wall, where the discrete absorption and emission of quanta by the atoms are leading to the thermal equilibrium [1, 2]. The quantum counterpart of this classical definition is the concept of an infinite potential well, ensuring a vanishing probability for the particle presence outside its surface.

The quantum version of the particle confinement is actually an eigenvalue problem, the discrete spectrum of the particle energies being a direct consequence

of the volume finiteness and of the shape of limiting surface. What is really quantized is the square of the linear momentum \vec{p}^2 and, through the intermediary of this quantity, the discrete values of the kinetic energy are obtained, resorting to the already known energy-momentum relationship of Relativistic Mechanics. So, the same Schrödinger-Helmholtz differential equation delivers the quantum states for either Planck radiation or rest-mass particles of given spin (bosons and fermions). The specific features of these different problems are materialized in polarization weight ($g_p = 2$) for mass-less particles, spin weight ($g_s = 2S + 1$) for bosons and fermions with rest mass, and in different statistical factors for the three kinds of particles (photons, rest mass bosons and rest mass fermions). The main characteristic of this kind of quantization is not the existence of a potential well (as in case of the Hydrogen atom), but rather the using of a finite region of the position space instead of the infinite three-dimensional Euclidian space. The cavity introduces a Dirichlet boundary condition. The history of these types of problems is very rich [3–7] and a good account of it was given by Gutierrez and Yanez [5].

The effect of the geometrical confinement upon the energy spectrum of the particles stored inside the cavity may be assigned to an additional quantizing, beyond that usually considered (referred to the discrete absorption and emission of quanta by the atoms of the cavity in view of reaching the thermal equilibrium). In this case, not only the energy exchanged between atoms and radiation is quantified, but also the radiation energy, or the gas kinetic energies, through the agency of the discrete spatial directions of the allowed wave-vectors. We named this quantum device as *double quantized cavity* (DQC) [8–11]. In physics literature, a good number of papers are describing the state statistics in quantum devices under the name of *quantum billiards* [12–15] and are relating them to the relatively new field of quantum chaos [15]. From this point of view, our study may be associated to a three-dimensional quantum billiard in a special double quantized regime in which thermodynamic aspects are however important.

The effect of the additional energy quantizing of ideal gases is strongly dependent on the box geometry and temperature, for low product of the atomic mass number, A_{at} and the adiabatic invariant, $TV^{2/3}$, *i.e.*, on $\gamma = A_{at}TV^{2/3}$. In the case of photons, this parameter has the form $\gamma = TV^{1/3}$. For small temperatures and small volumes, the energy density spectrum presents a discrete pattern (of lines with irregular intensities), strongly depending on the cavity geometry. In the energy spectrum, we found out locked states and states with negative radial quantum numbers, which have to be considered for an accurate description of the spherical cavity statistical physics. The total energy does not obey any longer the conventional laws, but a new ones, which include a corrective factor depending on γ and imposes a faster decrease to zero, for $\gamma \rightarrow 0$ [10, 11, 16].

1. SCHRÖDINGER-HELMHOLTZ EIGENVALUE PROBLEM IN A REFLECTING SPHERICAL CAVITY. KINETIC QUANTIZING

For a free particle of rest mass m_0 , the relativistic energy equation is written as

$$E^2 = (m_0 c^2)^2 + (c\vec{p})^2 \quad (1)$$

The particle energy equation is given by:

$$\vec{K}^2 - \frac{\vec{p}^2}{\hbar^2} = 0, \quad |\vec{K}| = K = \frac{E}{\hbar c} = \frac{2\pi}{\lambda} \quad (2)$$

Avoiding the spin effects and associating quantum operators to mechanical quantities, $p \rightarrow (\hbar/i)\nabla$, a Schrödinger-Helmholz (S-H) type equation is derived from eq. (2):

$$(\nabla^2 + K^2)\psi(\vec{r}) = 0, \quad (3)$$

Usually a Dirichlet boundary condition is taken as:

$$\psi_S = 0, \quad \text{for a finite-sized domain with defined geometry.} \quad (4)$$

In physics literature, the problem of distribution of the eigenvalues of Eq. (3), which are the cavity states (levels), is often treated in terms of the state density, ρ_ε , which is defined as the number of states (eigenvalues) lying around the energy ε in a (frequency) unit interval [2].

The number of states, Ω , inside the sphere volume $V = \frac{4}{3}\pi R^3$, coming from point like particles, whose linear momentum p satisfies the conditions $p \leq P$, $P = (h/2\pi)K$, K -quantized, is $\Omega = \frac{V}{h^3} \left(\frac{4}{3}\pi P^3 \right)$.

Let us introduce the dispersion relation:

$$\varepsilon = f(P)$$

and write down

$$\delta P = \frac{1}{f'(P)} \delta \varepsilon, \quad \delta \Omega = \frac{V}{h^3} \cdot \frac{4\pi P^2}{f'(P)} \delta \varepsilon,$$

where $\delta \Omega$ stands for all the states confined in the space volume V and having a kinetic energy $\varepsilon = f(P)$, in the range $f(P) - \frac{1}{2}\delta \varepsilon < \varepsilon < f(P) + \frac{1}{2}\delta \varepsilon$. Now, we define the quantity

$$\rho(P) = \frac{h}{V} \cdot \frac{\delta \Omega}{\delta \varepsilon} \equiv \frac{1}{h^2} \cdot \frac{4\pi P^2}{f'(P)}, \quad (5)$$

which has the following physical meaning: it is the number of states in the volume unity and frequency unity ($\delta v \equiv \delta \varepsilon / h$, defined conventionally, even for rest mass particles). For photons, one can write: $f(P) = cP$ and accordingly $\rho = K^2 / \pi c$. For unrelativistic particles, $f(P) = P^2 / 2m_0$ and accordingly, $\rho = 2m_0 K / h$.

Let us further consider the energy of all the particles confined in the volume V and having the energies in the proximity of $f(P)$:

$$\delta E = \frac{g_S}{F} V \rho(P) f(P) \frac{1}{h} \delta \varepsilon = \frac{g_S}{F} \frac{V}{h} \{ \varepsilon \rho(P) \} \delta \varepsilon. \quad (6)$$

Here a statistical factor g_S / F was introduced, for taking into account the establishing of the thermal equilibrium in compliance with the quantum statistical nature of the involved particles.

For particles, $g_S = 2S + 1$, $F_p = \exp \frac{\varepsilon - \mu}{kT} \pm 1$ and one obtains:

$$\delta E = \frac{2S + 1}{F_p} \cdot \frac{V}{\pi^2 \sqrt{2}} \cdot \left(\frac{\sqrt{\varepsilon m_0}}{\hbar} \right)^3 \delta \varepsilon. \quad (6a)$$

For quanta, $g_S = 2$, $F_q = \exp \frac{\varepsilon}{kT} - 1$, and one obtains:

$$\delta E = \frac{8\pi V}{(ch)^3} \frac{1}{F_q} \varepsilon^3 \delta \varepsilon; \quad (6b)$$

$$E = \int_0^\infty \frac{\delta E}{\delta \varepsilon} d\varepsilon = \sigma_P V T^4, \quad \sigma_P = \frac{8}{15} \pi^5 \frac{k^4}{c^3 h^3}.$$

Assuming that, through the agency of linear momentum P , the particle energy $\varepsilon(P)$ is quantized, one obtains the total energy inside the cavity as:

$$E = (2S + 1) \frac{V}{\pi^2 \sqrt{2}} \sum_{N=1}^\infty \frac{1}{F_p} \left(\frac{\sqrt{m_0 \varepsilon}}{\hbar} \right)^3 \frac{\delta \varepsilon}{\delta N} \quad (\text{for particles}) \quad (7a)$$

and

$$E = \frac{8\pi V}{(ch)^3} \sum_{N=1}^\infty \frac{1}{F_q} \varepsilon^3 \frac{\delta \varepsilon}{\delta N} \quad (\text{for quanta}). \quad (7b)$$

For a spherical cavity, $P \approx \frac{1}{4} h \left(\frac{N}{R} \right)$ and accordingly, one obtains:

$$\begin{cases} \varepsilon_q = cP = a_q N, & \frac{\delta \varepsilon_q}{\delta N} = a_q \equiv \frac{hc}{4R} \quad (\text{quanta}) \\ \varepsilon_P = \frac{1}{2m_0} P^2 = a_P N^2, & \frac{\delta \varepsilon_P}{\delta N} \equiv 2a_P N, \quad a_P \equiv \frac{h^2}{32m_0 R^2} \quad (\text{particles}) \end{cases}$$

and

$$\begin{cases} E_p = (2S+1) \sum_{N=1}^{\infty} g_p(N) \varepsilon_p(N) \frac{1}{F_p} \quad (\text{particles}) \\ E_q = 2 \sum_{N=1}^{\infty} g_q(N) \varepsilon_q(N) \frac{1}{F_q}, \quad (\text{quanta}) \end{cases} \quad (8)$$

Here, $g_p(N) = g_q(N) = \frac{\pi^2}{12} N^2$ is the level weight.

Well-known solving of S-H equation (3) in spherical coordinates leads to energy eigenfunctions with the form:

$$\begin{aligned} \psi(\vec{r}) &= U(\varphi, \vartheta) F(r), \\ \frac{d^2 F}{dr^2} + \frac{2}{r} \frac{dF}{dr} + \left[K^2 - \frac{l(l+1)}{r^2} \right] F(r) &= 0, \quad l = 0, 1, 2, \dots \end{aligned} \quad (9)$$

The Dirichlet boundary condition (at the reflecting walls) imposes:

$$F(R) = 0. \quad (10)$$

Together with the normalization condition for ψ , the boundary condition completely determines the energy eigenfunctions:

$$\begin{aligned} U(\varphi, \vartheta) &= \sqrt{4\pi} Y_{lm}(\vartheta, \varphi) \\ Y_{lm}(\vartheta, \varphi) &= \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos \vartheta) e^{im\varphi} \\ \int_0^{2\pi} d\varphi \int_0^{\pi} \sin \vartheta d\vartheta Y_{l'm'}^*(\vartheta, \varphi) Y_{lm}(\vartheta, \varphi) &= \delta_{ll'} \delta_{mm'} \\ F(r) &= F_{Nl}(r), \quad F_{Nl}(R) = 0, \\ F_{Nl}(r) &= \frac{1}{R\sqrt{2\pi r}} \frac{J_{l+1/2}(K_{Nl}r)}{J_{l+3/2}(K_{Nl}R)}, \\ \int_0^R F_{Nl}(r) F_{N'l}(r) 4\pi r^2 dr &= \delta_{NN'}, \quad \int_0^R 4\pi r^2 F_{Nl}^2(r) dr = 1. \end{aligned} \quad (11)$$

The following selection rules do hold:

$$\begin{aligned}
 n &= 0, 1, 2, \dots, \quad (\text{radial quantum number}), \\
 l &= 0, 1, 2, \dots, \quad (\text{angular quantum number}), \\
 m &= 0, \pm 1, \pm 2, \dots \pm l, \quad (\text{magnetic quantum number}), \\
 N &= 2n + l = 1, 2, 3, \dots; \quad n \text{ and } l \text{ cannot be zero simultaneously.}
 \end{aligned} \tag{12}$$

The name of the eigenfunction indices is usually taken by analogy to their quantum numbers of the hydrogen atom. These selection rules define an additional quantisation of the cavity, leading to allowed states defined by two quantum numbers (N, l) and degeneracy. Thus, boundary condition (10) and ortho-normalization introduce a complex and discrete energy spectrum interpretable in terms of a multiplet structure and a non-degenerated fine structure. We shall call the spherical cavities with a small number of states as *double quantised spherical cavities* (DQSC).

The asymptotic eigenfunctions have the form:

$$F_{Nl} \rightarrow F_{Nl}^0 = (2\pi R)^{-\frac{1}{2}} \cdot \frac{\sin\left(K_{Nl}r - l\frac{\pi}{2}\right)}{r} \tag{13}$$

$$F_{Nl}^0(R) = 0 \rightarrow K_{Nl} \cdot R = N\frac{\pi}{2}, \quad N = 2n + l,$$

$$l = 2p, \quad N = 2q: \quad F_{Nl}(r) = +\frac{(-1)^p}{r\sqrt{2\pi R}} \sin\left(\frac{\pi}{2}N \cdot \frac{r}{R}\right) \tag{13a}$$

$$l = 2p + 1, \quad N = 2q + 1: \quad F_{Nl}(r) = -\frac{(-1)^p}{r\sqrt{2\pi R}} \cos\left(\frac{\pi}{2}N \cdot \frac{r}{R}\right). \tag{13b}$$

The total energy inside the cavity is:

$$E = \sum \frac{1}{F_p} 2g_N E_N \Delta N, \tag{14}$$

$$\text{with: } z \equiv \frac{\pi}{2}N, \quad E_N = \frac{c\hbar}{R}z, \quad \frac{E_N}{kT} = \frac{ch}{2\pi R} \cdot \frac{z}{kT} = az. \tag{15}$$

The eigenfunctions, $F_{Nl}^0(r)$, make up an ortho-normalized set of functions, over the discrete set of quantum numbers $N = 1, 2, 3, \dots \infty$. This feature entitles us not only to adopt the asymptotic classifying of states for the real ones, but also to use asymptotic wave functions for calculating energies and weights.

This adoption fails to hold exactly for DQSC at low temperatures, a case when a *rigorous treatment of the quantizing problem* must be applied. The states of

DQSC depend on two quantum numbers, which cannot be grouped exactly for delivering a single quantum number of energy, as in the case of cubic cavity (DQCC). The dependence of the energy states on a single quantum number, N , may however be considered as a good approximation if all components of the multiplet are replaced by a single energy level, whose position is almost coincident with the multiplet gravity center and whose weight is the sum of all (allowed) components. In this approximation, N may be considered as a good energy quantum number (playing similar role as the principal quantum number of the Hydrogen atom).

The quantity $z_{NI} = K_{NI}R = \frac{R}{\hbar}\sqrt{\bar{p}_{NI}^2}$ is quantized according to eqs. (2)–(3).

The results of the numerical calculations are listed against the increasing values of this quantity z_{NI} and the magnetic weight $g_{magnetic} = (2l+1)$ is associated to each state (N, l) . For large quantum numbers N , it is ascertained the asymptotic rule $(2/\pi)z_{NI} \approx N$. We illustrate in Annex 2, with some numerical results for moderately high quantum numbers, N , that indeed:

$$\bar{z} = \frac{\sum g_{rel.z}}{\sum g_{rel.}} \approx \frac{\pi}{2}N. \quad (16)$$

For quanta (photons, gravitons, neutrinos, etc.), the discrete energies are:

$$E_{NI} = c\sqrt{\bar{p}_{NI}^2} = \frac{c\hbar}{R}z_{NI} \approx \frac{ch}{4R}N = h\nu_0N, \quad \nu_0 = \frac{c}{4R}, \quad (17.a)$$

which leads to an asymptotically equidistant spectrum resembling that delivered by the harmonic oscillator.

For (non-relativistic) rest mass particles, the discrete energies are:

$$E_{NI} = \frac{1}{2m_0}\bar{p}_{NI}^2 = \frac{1}{2m_0}\left(\frac{\hbar}{R}z_{NI}\right)^2 \approx \frac{h^2}{32m_0R^2}N^2. \quad (17.b)$$

For the first 33 states (*i.e.* for $z \leq 17.22$), all the three quantum numbers (N, l, n) are positive quantities and the analogy with the Hydrogen atom may be extended for evaluating the weights

$$g(N) = \sum_{l \leq N} (2l+1) = \frac{1}{2}(N+1)(N+2), \quad (18)$$

and

$$g_{total} = \begin{cases} 2g(N) \approx N^2 & \text{(for quanta)} \\ (2S+1)g(N) \approx (2S+1)\frac{1}{2}N^2 & \text{(for particles)} \end{cases}. \quad (19)$$

For spectra which extend over the limit $z = 17.22$ (either quanta or particles), it is necessary to take into account the existence of states with negative radial

quantum number, $n < 0$. Accordingly, the weights should be calculated by resorting to the state diagram from Fig. 1:

$$g(N) = \sum_{l \leq \frac{\pi}{\sqrt{6}}N} (2l+1) = \frac{1}{2} \left(\frac{\pi}{\sqrt{6}} + 1 \right) \left(\frac{\pi}{\sqrt{6}} + 2 \right) \approx \frac{\pi^2}{6} \left(\frac{1}{2} N^2 \right)$$

and

$$g_{total} = \begin{cases} 2g(N) \approx \frac{\pi^2}{6} N^2 & \text{(quanta)} \\ (2S+1)g(N) \approx (2S+1) \frac{\pi^2}{12} N^2 & \text{(particles)} \end{cases} \quad (20)$$

The presence of the factor $(\pi^2/6)$ ensures the convergence of the energy distribution of quanta towards the Planck limit.

Unlike the Planck spectrum of a cubic cavity, which is made up of lines depending of a single quantum number N ($N = n_1^2 + n_2^2 + n_3^2$), the Planck spectrum in a spherical cavity is made up of a series of fine structure multiplets (N, l) . As previously discussed, the multiplet is degenerated only in the asymptotic limit $R \rightarrow \infty$. For a finite volume, the components of a certain multiplet n are different one from another and randomly spread out in the proximity of $z = (\pi/2)N$ with a half width of $\pm(\pi/2)$. A good approximation is obtained by associating to the whole multiplet N a linear momentum, $P = \frac{1}{4}h\left(\frac{N}{R}\right)$.

2. MORE ACCURATE EXPRESSION FOR THE STATE WEIGHTS IN SPHERICAL CAVITY

We noticed that the exact weights, presumable of the form:

$$g(N) = \left(\frac{\pi^2}{12} N^2 + c_1 N + c_2 \right) \quad (21)$$

cannot be further determined through asymptotic methods (for $N \rightarrow \infty$). However, the analysis of the multiplet structure of spectra may deliver information on the coefficients c_1 and c_2 . For instance, from the weight of the energy interval,

$$\Delta g \approx \frac{2}{3\pi} (z - 1.3223)^2 \cdot \Delta z \quad (22)$$

and the relations $z \approx (\pi/2)N$, $\Delta N = 1$, $\Delta g = g(N)$, we may derive a better expression of the asymptotic weight:

$$g(N) \approx \frac{\pi^2}{12}(N - 0.8418)^2. \quad (23)$$

By comparing the two alternative expressions of $g(N)$, we conclude that $c_1 < 0$. This may be considered as a consequence of the existence of the locked states (the black points in the state diagram from Fig. 1. The **locked states** (called equally antiresonances, *e.g.* in [11, 16], or dark states) are states complying with the asymptotic energy formula, but not solutions of equation (3).

The lowest state, $N = 1$, has a single fine structure component, which is locked. This is equivalent to state that the real ground state is not (1, 1) but (2, 0). All the multiplets of the spectrum, having $N > 1$, may contain a certain number of locked components. The presence of such locked sub-states in the framework of a multiplet state N , does not alter the average energy of the multiplet, but decreases, to a certain extent, the weight of the respective multiplet. The Planck limit of the spectrum is not altered, but only the convergence speed toward this limit. For instance, the presence of the locked sub-state ($N = 7, l = 5$) in the multiplet $N = 7$ decreases the weight of the state from $g(7) = 36$ to $g(7) = 25$, *i.e.* with 30.56%. This means that, unlike the case of the cubic cavity, in a spherical cavity, we have never a drastically locked energy.

A more reliable correction to $g(N)$ is obtained using a best fit of the integral weights for $N \leq 32$:

$$g(N) \approx \frac{\pi^2}{12}(N - 0.2421053)^2, \quad (24)$$

which gives again a negative value of c_1 in agreement with the existence of the locked states.

We may suppose that, as far as $N \rightarrow \infty$, the contributions to the coefficient c_1 coming from the so called ‘‘surface terms’’ [10] and those coming from locked states reciprocally cancel to a great extent and the simple approximation $g(N) \approx \frac{\pi^2}{12}N^2$ turns out to be a good one.

A more accurate approximation of $g(N)$ is obtained by studying the systematic of the multiplet pairs, $\psi(N) = g(N - 2) + g(N - 1)$. A best fit of $\psi(N)$ (shown in Annex 3) is:

$$\psi(N) = \frac{\pi^2}{6}(N^2 - 3.4842106N + 3.3682642), \quad (25)$$

which leads to

$$g(N) \approx \frac{\pi^2}{12}(N^2 - 0.4842106N + 0.1419483). \quad (26)$$

This means that the locked state effect slightly exceeds the surface effect. Formula (26) is in good agreement with the calculated data of the multiplet weights.

3. CHARACTERISTICS OF THE ENERGY SPECTRUM OF SPHERICAL CAVITY

Concerning the energy spectrum problem, we may point out the following specific features:

1) the spectrum is a discrete one, the density of lines, on a frequency unit being inversely proportional to the radius of spherical cavity;

2) the spectrum – analyzed in terms of the radial quantum number, n and orbital quantum number, l , in (n, l) states, having a $(2l + 1)$ – magnetic type degeneracy – is a succession of higher and higher degenerated multiplets (this second degeneracy being however not intrinsic to the formalism, but rather determined by the classification error associated to replacing of the exact wave function F_{Nl} with the asymptotic ones $F_{Nl}^{(0)}$);

3) owing to the peculiarity that (n, l) are combined one to another to make up the principal quantum number, N , in a specific way $2n + l = N$, (different from the homologous way we come across in the hydrogen atom case $n + l = N$), two adjacent states, inside a multiplet labeled by N , namely (N, l) , (N, l') , are placed at a relative distance $|l - l'| = 2$. At the same time, N and l always have the same parity, whence two adjacent states (N, l) , (N', l) are placed at a relative distance $|N - N'| = 2$.

These are serious arguments to classify the states not in the (n, l) plane, but rather in the (N, l) plane, where the energy states cover the region between the vertical axis Ol and the bisectrix of the plane NOl , presenting the characteristic aspects of a chess table (Fig. 1). Reading this diagram vertically, the multiplet structure of the spectrum is revealed: $N = 1$ singlet, $N = 2$ doublet, $N = 3$ doublet, $N = 4$ triplet, $N = 5$ triplet, $N = 6$ quadruplet, $N = 7$ quadruplet etc. On the other hand, all states with the same l , which appear along the corresponding horizontal line, have the same degeneracy, $g = 2l + 1$.

By going over from the asymptotic radial wave functions to the exact ones, the following effects occur:

- 1) the energy state positions are displaced;
- 2) the degeneracy of the multiplets is removed;
- 3) the multiplet with a quantum number N is penetrating the following multiplet with the quantum number $N + 1$, in such a way that the components of these multiplets occupy the same frequency interval, their succession being apparently random [for example, the multiplets $(30, l)$ and $(31, l)$ are interlaced and have very close weight centers, *i.e.* $N_{30} = 30.8670$ and $N_{31} = 30.9227$];
- 4) some of the asymptotic states (N, l) became forbidden (**locked states**, the black cells in Fig. 1) [10, 11];

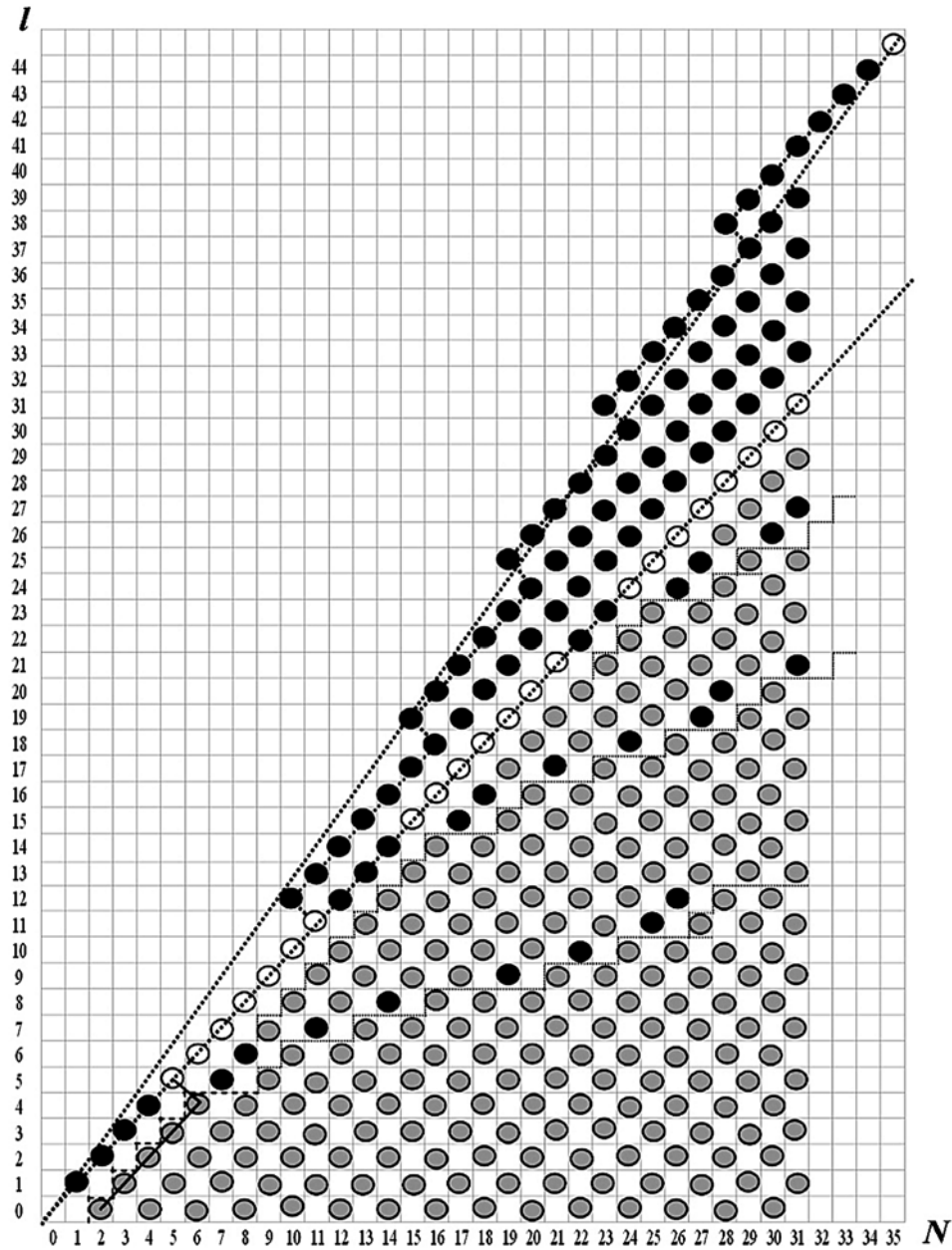


Fig. 1. – The diagram of the lowest energy states of discrete spectrum in the spherical cavity in the asymptotic approximation can be described in terms of a (l, N) “chess-board”. Allowed states are drawn in gray and labeled by the $(2l + 1)$ -“magnetic” weights. The ground state is the singlet $N = 1$, ($n = 0$, $l = 1$). Reading the diagram vertically, the multiplet structure of the spectrum is revealed. The black cells correspond to the locked states (antiresonances). The dark gray cells correspond to the states with negative radial quantum number ($n < 0$) and occur above the diagonal ($l = N$).

- 5) there are some **states with negative radial quantum number** (the dark gray cells in Fig. 1, marked by an asterisk in the list in Annex 1 and occurring for $N \geq 10$), which yield the factor $\pi^2/6$ in the total degeneracy;
- 6) in the discrete spectrum, the forbidden states (under the diagonal of the (NOI) quadrant) occur earlier than states with negative radial quantum number and together with the pairs of multiplets structured as mentioned at the point 4), seems to produce the maximum deviations of discrete spectrum from the classical one.

The identification of the allowed states are done with the following procedure:

α) given l , from the condition $J_{l+(1/2)}(z) = 0$, we obtain a the discrete series of values: $z_p(l), z_1(l), z_2(l), \dots, z_p(l) \dots$;

β) for every $z_p(l)$, there is an integer positive number, N , which satisfies the inequalities:

$$(N-1) < (2/\pi) \cdot z_p(l) < (N+1) \quad (27)$$

and which has the same parity as l . The state ($l = 0, N = 0$) does not exist. The first (lowest) states (N, l) are: (2,0), (3,1), (4,2), (4,0), (5,3) etc.

γ) The states are ordering by the increasing series of values of the number $z_k = 2\pi(\nu_{nl}R/c)$ and are characterized by the weight g_k of the magnetic degeneracy. (E.g.: the 21-st state is found for $z_{21} = 13.915823$ and $g_{21} = 19$ leading to: $2l + 1 = 19$, from which $l = 9$; then, a number N which has the same parity as l (in this case, odd) and which satisfies the ineq. (27), $(N-1) < 8.859088 < (N+1)$, is found as $N = 9$; thus, the 21-st state is (9,9) and it has: $n = (N-l)/2 = 0$.

We notice that the first state with **negative radial number** ($n < 0$) is the 34-th one, which is characterized by: $N = 10, l = 12, g_{34} = 25, n = -1$. The negative radial numbers increase in absolute value with the order number of the state, as shown in Fig. 1 and in the Annex 4. In the chessboard diagram (N, l), the negative radial numbers are found between the lines $l = N$ and $l = (\pi/\sqrt{6})N$ (the superior asymptotic border of the allowed states).

4. PLANCK RADIATION IN A REFLECTING SPHERICAL CAVITY WITH RELATIVELY SMALL NUMBER OF STATES

We have shown that the intervals between the successive states in DQSC have a random distribution around a value, which decrease with the frequency (z_k) increasing. The corresponding weights ($2l_k + 1$) are also randomly distributed. These two characteristics of the energy levels in DQSC lead to a discrete Planck

spectrum of extreme complexity. The fact that the principal quantum number, N , is not proportional to z outside the asymptotic limit (*e.g.* the state 9, with the quantum numbers (5,5), follows the state (6,2)) further complicates the problem. However, even in this case, one can establish some statistical rules and arrive, for photons, to the Planck radiation law.

In order to force an analogy with Planck radiation of a cubic cavity, we can reduce formally the frequency spectrum of spherical cavity to an **equivalent multi-channel spectrum**, depending solely on a single integer parameter, namely *the number, k , of the equidistant frequency channel* of a spectrometric analyzer. We can firstly adopt equidistant channels with a conventionally fixed width $\Delta z = 0.5$.

The energy of the Planck radiation in the spherical cavity may be written as

$$E_P = \sum \frac{1}{F_P} 2E(z) \Delta g(z) \quad (28)$$

$$E(z) = azkT, \quad a = \frac{ch}{2\pi k} (RT)^{-1}$$

$$F_P = \exp(az) - 1$$

$$\Delta g(z) = \zeta \frac{2z^2}{2\pi} \Delta z, \quad \zeta \equiv \frac{3\pi}{2z^2} \frac{\Delta g(z)}{\Delta z}, \quad \bar{\zeta} = 1 \quad (29)$$

We first check that the quantity ζ does change randomly from a channel to another, having the average value 1. Here $\Delta g(z)$ is the sum of all the weights inside a given channel having the width, Δz , centered in z .

Thereafter, the sum

$$E_P = \sum_{(z)} \zeta \frac{2azkT}{\exp(az) - 1} \cdot \frac{2z^2}{3\pi} \Delta z \quad (30)$$

is approximated by an integration, provided that $a\Delta z = \Delta(az) \ll 1$,

$$E_P = \frac{4}{3\pi} \cdot \frac{kT}{a^3} \int_0^\infty \frac{\xi^3 d\xi}{\exp \xi - 1} = \sigma_P VT^4, \quad (31)$$

$$\sigma_P = \frac{8}{15} \pi^5 \frac{k^4}{c^3 h^3} = 7.5659121 \times 10^{-15} \text{ (erg)/(K}^4\text{cm}^3\text{)}.$$

In a given channel, $z(k) = \left(k - \frac{1}{2}\right) \Delta z$, the total weight, of the levels stored inside, may pertain to two adjacent multiplets $(N, N+1)$ but not more. This is a

consequence of the fact that the components of a multiplet N have a maximal dispersion of $(\Delta z)_{Max} = \pi$.

We can truncate the Planck spectrum of DQSC up to a highest significant state, which would bring a contribution smaller than 10^{-2} to the spectrum energy and can be calculated as:

$$z_T \geq 48 RT. \quad (32)$$

This truncation frequency of the Planck spectrum of DQSC can be used as in the case of the equivalent cubic cavity. From Fig. 2a,b,c, one can see that this relation holds well; in Fig. 2d, for $RT = 1.5$, the frequency interval covered by the 300 Bessel roots ($z_T = 50$) is not long enough to ensure an error of 10^{-2} , which proves again that (32) is correct.

One can remark also, from Fig. 2a, that the maximal condition, still preserving the validity of summing up to the 70th state, could be a sphere of 1 cm diameter kept at an absolute temperature of 1 K. This means that we can expect *quantum effects in the spherical black-body radiation energy for macroscopic conditions*.

Assuming that the highest significant state reaches the asymptotic limit, one can take:

$$N_T \approx (2/\pi)z_T \approx 31 RT, \quad (33)$$

which is useful for the asymptotic formalism.

We define the energy density on the frequency unit as

$$u(\nu) = \frac{h}{V} \cdot \frac{\Delta E_P}{\Delta E_q}, \quad (34)$$

where: $\Delta E_P = \frac{4}{3\pi} kT \zeta \frac{z^3}{\exp az - 1} a \Delta z$.

We have taken: $\zeta \equiv \frac{3}{2} \pi \cdot \frac{\Delta g}{z^2 \Delta z}$, $\bar{\zeta} = 1$, $u(\nu = f(z))$. It is important to remark again that the factor ζ exhibits a random fluctuation around the value $\zeta = 1$.

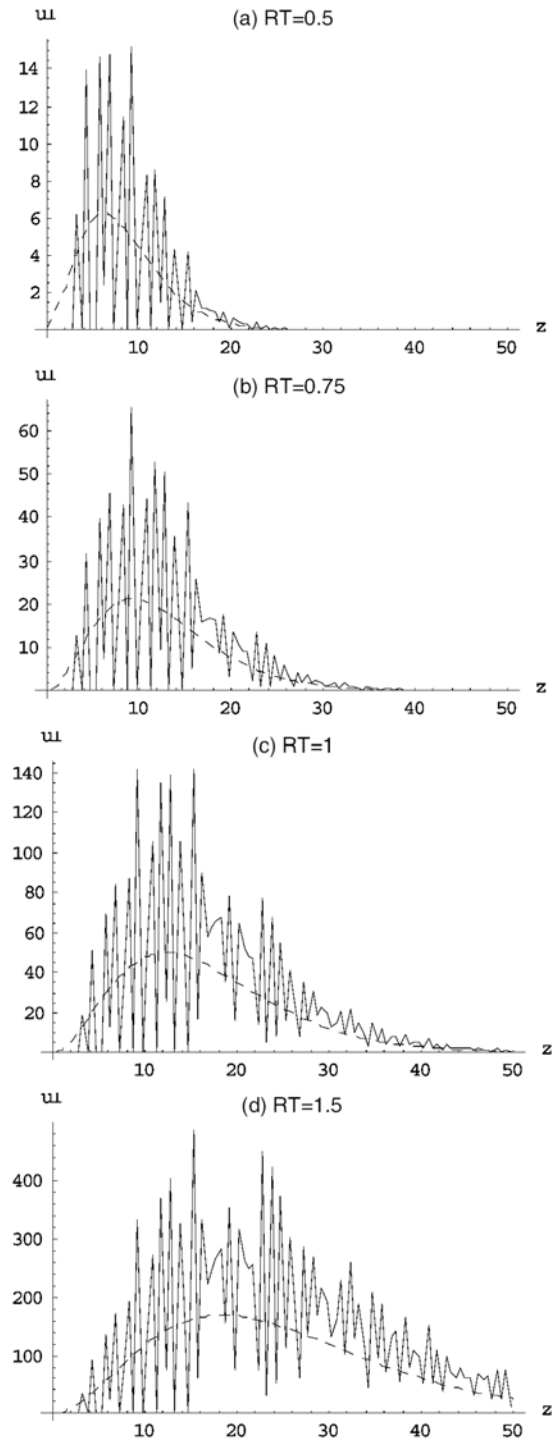
$$\frac{4\pi R^3}{3} \frac{f(z)}{h} = \frac{4}{3\pi} \cdot \frac{z^3}{\exp az - 1} \zeta, \quad (35)$$

with $a = \frac{\alpha_1}{\gamma_1}$, $\alpha_1 = \frac{hc}{2\pi k}$, $\gamma_1 = RT \rightarrow a = \frac{hc}{2\pi kRT}$.

$\varepsilon_q = az \cdot kT = \frac{hc}{2\pi} \left(\frac{z}{R} \right)$ is the energy of the quantum.

The Planck spectrum of the spherical cavity is to a great extent similar to that given by a cubic cavity (the real spectrum exhibits a random fluctuation around the Planck smoothed curve). Numerical calculations were performed for $\Delta z = 0.5$ and $RT = 0.50; 0.75; 1.0$ and 1.5 cm·K. By ensuring the same experimental conditions

Fig. 2. – The discrete Planck spectrum of DQSC plotted against the dimensionless frequency z (joined solid lines) and compared to the continuous Planck spectrum (dashed line) for several values of the adiabatic invariant, RT : (a) 0.5; (b) 0.75; (c) 1; (d) 1.5 cm·K.



(cavities of equal volumes and channel type registration of equal energy bands) the spectrum coming from a spherical cavity resembles the most with that of a cubic cavity for the same temperature.

For $\Delta z = 0.5$ and $40 \leq k \leq 60$ (i.e. $19.75 < z < 29.75$), a best fit delivers the approximation: $\Delta g \approx \frac{2}{3\pi}(z-1.3223)^2 \Delta z$, which is in good agreement with the asymptotic result $\bar{\zeta} = 1$.

The same *reciprocity rule* (as for DQCC) holds: the cavity size and the temperature are reciprocal parameters in the DQSC, i.e. the same effects (in the thermodynamics of the photon gas) can be obtained either by varying R or by varying T , if their product remains constant.

From the view point of reaching the Planck radiation formula, the method of energy channels (which integrates over the multiplet structure of the spectrum) is equivalent to the explicit consideration of multiplet structure of spectrum in spherical cavity (including the locked states and states with negative radial numbers), as shows the subsequent calculation:

$$\varepsilon_N = Nh\nu_0, \quad \nu_0 = \frac{c}{4R}, \quad V = \frac{4}{3}\pi R^3, \quad L \approx \frac{\pi}{\sqrt{6}}N, \quad (36)$$

$$g(N) = 2 \cdot \frac{1}{2}(L+1)(L+2) \approx \frac{\pi^2}{6}N^2 \quad (37)$$

The summation over all multiplet structure of the spectrum:

$$E = \sum_{N=1}^{\infty} \frac{\varepsilon_N \cdot g(N)}{\exp\left(\frac{\varepsilon_N}{kT}\right) - 1} = \frac{\pi^2}{6}h\nu_0 \sum_{N=1}^{\infty} \frac{N^3}{e^{aN} - 1} \quad \text{with} \quad a = \frac{ch}{4kRT} \quad (38)$$

takes, in the asymptotic limit, the well-known form:

$$E \rightarrow \frac{\pi^2}{6} \frac{h\nu_0}{a^4} \int_0^{\infty} \frac{u^3 du}{e^u - 1} = \sigma_p \cdot V \cdot T^4 \quad \text{with} \quad \sigma_p = \frac{8}{15}\pi^5 \cdot \frac{k^4}{h^3 \cdot c^3}. \quad (39)$$

5. THE ENERGY SPECTRUM OF QUANTUM GAS WITH FREE PARTICLES WITH REST MASS IN A REFLECTING SPHERICAL CAVITY

The energy of the gas is written as:

$$E = \sum_{N=1}^{\infty} \frac{\varepsilon_N \cdot g(S) \cdot g(N)}{\exp\left(-\frac{\mu}{kT}\right) \cdot \exp\left(\frac{\varepsilon_N}{kT}\right) \pm 1}, \quad (40)$$

where the negative quantum numbers n are included, but the locked states (antiresonances) are ignored. Here:

$$g(S) = 2S + 1, \quad g(N) = \frac{1}{2}(L+1)(L+2)$$

$$L \approx \frac{\pi}{\sqrt{6}}N, \quad \rightarrow g(N) \approx \frac{\pi^2}{12}N^2 \quad (\text{for } N \gg 1). \quad (41)$$

We consider that the particle energies are not relativistic

$$\varepsilon_N = \frac{1}{2m_0} \vec{p}_N^2 \rightarrow \varepsilon_N = \eta \left(\frac{N}{R} \right)^2, \quad \eta = \frac{h^2}{32m_0} \quad (42)$$

and the number of multiplets N is relatively large

$$g(N) \approx \frac{\pi^2}{12} \cdot \frac{\varepsilon_N}{\eta} R^2, \quad \exp\left(-\frac{\mu}{kT}\right) = \frac{1}{A}. \quad (43)$$

In this case, we can obtain:

$$E = \sum_{N=1}^{\infty} \frac{(2S+1) \cdot \frac{\pi^2}{12} \cdot \frac{\varepsilon_N^2}{\eta} R^2 \cdot \varepsilon_N}{\frac{1}{A} \cdot \exp\left(\frac{\varepsilon_N}{kT}\right) \pm 1}; \quad (44)$$

$$N = R \cdot \frac{\varepsilon_N^{1/2}}{\sqrt{\eta}} \rightarrow \delta N = \frac{R \delta \varepsilon_N}{2\sqrt{\eta \varepsilon_N}} \equiv 1. \quad (45)$$

Thus, the exact formula of the quantum gas in the classical limit can be found:

$$E = \sum_{N=1}^{\infty} \frac{\frac{\pi^2}{24} (2S+1) R^3 \left(\frac{\varepsilon_N}{\eta} \right)^{3/2} \delta \varepsilon_N}{\frac{1}{A} \cdot \exp\left(\frac{\varepsilon_N}{kT}\right) \pm 1} =$$

$$= \frac{1}{\pi^2 \sqrt{2}} (2S+1) \frac{m_0^{3/2}}{\hbar^3} \left(\frac{4}{3} \pi R^3 \right) \sum_{N=1}^{\infty} \frac{\varepsilon_N^{3/2} \cdot \delta \varepsilon_N}{\frac{1}{A} \cdot \exp\left(\frac{\varepsilon_N}{kT}\right) \pm 1} \quad (46)$$

6. CONCLUSIONS

We entered upon the quantization of the kinetic energy inside the spherical cavity by assuming a close analogy with the quantum problem of the Hydrogen atom. However, this analogy does hold well only for $N < 10$, case when no negative quantum number is present.

The radial quantum number, $n = (N - l)/2$, associated to the square momentum quantization inside an ideal spherical cavity, necessarily have some negative values for $N \geq 10$. In the diagram ($x = N$, $y = l$), these states are asymptotically confined between the right lines $y = x$ and $y = (\pi/\sqrt{6})x$. The maximal negative value of n for a level pertaining to a multiplet N is

$$-\left(\frac{\pi}{\sqrt{6}} - 1\right)\frac{1}{2}N = -0.1412749N.$$

The main argument, for considering the states with $n < 0$ as physical ones, is the fact that they are solutions of the S-H equations with positive eigenvalues for the operator \vec{p}^2 . Another valuable argument is that the statistical distributions of the Planck quanta and of fermions and bosons of rest-mass particles cannot be obtained if these states would be disregarded.

The states with $n < 0$, id est with $l > N$ are, on a side, necessary for reaching the statistical equilibrium, but, at the same time, they are strange enough for their content of rotational energy. This peculiar aspect may be explained in the frame work of a kinetic quantization only by assuming the existence of a certain correlation implying more than single-particle states.

The discrete pattern of energy Planck spectrum in a spherical cavity is a succession of fine-structure non-degenerated multiplets N exhibiting few characteristic features: 1) The quantized $|\vec{p}|$ is almost proportional to N ; 2) the fine structure components of the multiplet N are spread out over on interval $\Delta z = \pi$, centered in $z = (\pi/2)N$. The baricenter of the multiplet N is $\bar{z}_N = (\pi/2)N$; 3) There is a continuous interlacing of a multiplets so that in the interval $z \in [(N-1)\pi/2, N\pi/2]$ of a multiplet N , we come across components of the multiplet $(N-1)$, while in the interval $[N\pi/2, (N+1)\pi/2]$ of the same multiplet N , we come across components pertaining to the multiplet $(N+1)$.

As a result of the cancellation of the so called surface effects with the locked states effect, the weight $g(N)$ of the multiplet N may be written, within a good approximation, as $g(N) \approx (\pi^2/12)N^2 + c_1N$, $c_1 < 0$. The multiplet structure of $|\vec{p}|$ -quantized spectrum in the spherical cavity is a common basis for entering upon the quantum gases and/or Planck radiation at small volumes and temperatures.

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ANNEX 1

The allowed states of DQSC in the interval $z \in (0, 20)$

| k | (N, l) | g_k | $z_k = 2\pi v_n R/c$ |
|-----|----------|-------|----------------------|
| 1 | (2, 0) | 1 | 3.141 592 7 |
| 2 | (3, 1) | 3 | 4.493 409 5 |
| 3 | (4, 2) | 5 | 5.763 459 2 |
| 4 | (4, 0) | 1 | 6.283 185 3 |
| 5 | (5, 3) | 7 | 6.987 932 0 |
| 6 | (5, 1) | 3 | 7.725 251 8 |
| 7 | (6, 4) | 9 | 8.182 561 6 |
| 8 | (6, 2) | 5 | 9.095 011 3 |
| 9 | (5, 5) | 11 | 9.355 812 0 |
| 10 | (6, 0) | 1 | 9.424 778 0 |
| 11 | (7, 3) | 7 | 10.417 119 0 |
| 12 | (6, 6) | 13 | 10.512 836 0 |
| 13 | (7, 1) | 3 | 10.904 122 0 |
| 14 | (7, 7) | 15 | 11.657 033 0 |
| 15 | (8, 4) | 9 | 11.704 908 0 |
| 16 | (8, 2) | 5 | 12.322 941 0 |

(continues)

ANNEX 1 (continued)

| k | (N, l) | g_k | $z_k = 2\pi\nu_n R/c$ |
|-----|----------|-------|-----------------------|
| 17 | (8, 0) | 1 | 12.566 371 0 |
| 18 | (8, 8) | 17 | 12.790 718 2 |
| 19 | (9, 5) | 11 | 12.966 531 0 |
| 20 | (9, 3) | 7 | 13.698 023 0 |
| 21 | (9, 9) | 19 | 13.915 823 0 |
| 22 | (9, 1) | 3 | 14.066 194 0 |
| 23 | (10, 6) | 13 | 14.207 393 0 |
| 24 | (10,10) | 21 | 15.033 469 3 |
| 25 | (10, 4) | 9 | 15.039 665 0 |
| 26 | (9, 7) | 15 | 15.431 288 0 |
| 27 | (10, 2) | 5 | 15.514 603 0 |
| 28 | (10, 0) | 1 | 15.707 963 0 |
| 29 | (11, 11) | 23 | 16.144 741 0 |
| 30 | (11, 5) | 11 | 16.354 710 0 |
| 31 | (10, 8) | 17 | 16.641 003 0 |
| 32 | (11, 3) | 7 | 16.923 622 0 |
| 33 | (11, 1) | 3 | 17.220 755 0 |
| 34 | (10,12) | 25 | 17.250 454 8* |
| 35 | (12, 6) | 13 | 17.647 975 0 |
| 36 | (11, 9) | 19 | 17.838 644 0 |
| 37 | (12, 4) | 9 | 18.301 256 0 |
| 38 | (11,13) | 27 | 18.351 261 2* |
| 39 | (12, 2) | 5 | 18.689 036 0 |
| 40 | (12, 0) | 1 | 18.849 556 0 |
| 41 | (13,7) | 15 | 18.922 999 2 |
| 42 | (12, 10) | 21 | 19.025 853 5 |
| 43 | (12,14) | 29 | 19.447 701 5* |
| 44 | (13,11) | 11 | 19.653 152 1 |

ANNEX 2

Replacing the multiplets by single energy levels positioned in the multiplet baricenter and weighted by the sum of the component weights

| Multiplet quantum number, N | Weight, $g(N)$ | Number of components | $\frac{\sqrt{12}}{\pi} \cdot \frac{g^{1/2}(N)}{N}$ |
|-------------------------------|----------------|----------------------|--|
| 5 | 21 | 3 | 1.010 603 |
| 10 | 91 | 7 | 1.051 868 |
| 15 | 210 | 10 | 1.065 269 |
| 20 | 378 | 14 | 1.071 906 |
| 25 | 532 | 16 | 1.017 318 |
| 30 | 731 | 19 | 0.993 752 |

For the multiplet $N = 19$, we can calculate:

| l | $g(l)$ | z | $g_{rel.}(\%)$ |
|-----|--------|---------------------|----------------|
| 15 | 31 | 28.6974 | 10.72 664 |
| 7 | 15 | 28.8704 | 5.19 031 |
| 23 | 47 | 29.1756 | 16.26 298 |
| 5 | 11 | 29.3326 | 3.80 623 |
| 19 | 39 | 29.3971 | 13.49 481 |
| 3 | 7 | 29.6426 | 2.42 215 |
| 1 | 3 | 29.8116 | 1.03 806 |
| 13 | 27 | 29.8893 -baricenter | 9.34 256 |
| 11 | 23 | 30.8208 | 7.95 848 |
| 17 | 35 | 31.0624 | 12.11 073 |
| 25 | 51 | 31.3127 | 17.64 706 |
| 9 | 19 | missing component | |
| 21 | 43 | missing component | |

$$\sum g = 289; \quad \sum g_{rel.} = 100.00001; \quad \bar{z} = \frac{\sum g_{rel.} z}{\sum g_{rel.}} = 29.965498 \text{ is close to } \frac{\pi}{2} N = 29.84513 \text{ whence}$$

$$\bar{z} \approx \frac{\pi}{2} N.$$

The multiplet $N = 20$:

| l | $g(l)$ | z | $g_{rel.}(\%)$ |
|-----|--------|---------------------|----------------|
| 16 | 33 | 29.8828 | 8.73 016 |
| 8 | 17 | 30.2173 | 4.49 753 |
| 24 | 49 | 30.2450 | 12.96 296 |
| 20 | 41 | 30.5251 | 10.84 656 |
| 6 | 13 | 30.7304 | 3.43 915 |
| 4 | 9 | 31.0939 | 2.38 095 |
| 14 | 29 | 31.1206 | 7.67 196 |
| 2 | 5 | 31.3201 | 1.32 275 |
| 0 | 1 | 31.4159 -baricenter | 0.26 455 |
| 12 | 25 | 32.0967 | 6.61 376 |
| 18 | 37 | 32.2366 | 9.78 836 |
| 26 | 53 | 32.3787 | 14.02 116 |
| 22 | 45 | 32.7708 | 11.90 476 |
| 10 | 21 | 32.8705 | 5.55 556 |

$$\sum g = 378; \quad \sum g_{rel.} = 100.00017; \quad \bar{z} = \frac{\sum g_{rel.} z}{\sum g_{rel.}} = 31.427099 \text{ is close to } \frac{\pi}{2} N = 31.41593 \text{ whence}$$

$$\text{again: } \bar{z} \approx \frac{\pi}{2} N.$$

ANNEX 3

Calculation of the multiplet pairs for obtaining an accurate approximation for the state weight

| N | $\Psi(N)$ calculated | $\Psi(N)$ best fit |
|-----|----------------------|--------------------|
| 4 | 9 | 8.93 |
| 6 | 36 | 30.37 |
| 8 | 53 | 64.97 |
| 10 | 121 | 112.72 |
| 12 | 160 | 173.64 |
| 14 | 262 | 247.71 |
| 16 | 367 | 334.94 |
| 18 | 434 | 435.34 |
| 20 | 532 | 548.89 |
| 22 | 636 | 675.60 |
| 24 | 865 | 815.47 |
| 26 | 979 | 968.50 |
| 28 | 1050 | 1134.69 |
| 30 | 1342 | 1314.04 |
| 32 | 1506 | 1506.55 |

ANNEX 4

Some states with negative radial numbers (and large N)

| State number | (N, l) | n | Parity |
|--------------|----------|-----|--------|
| 233 | (29,37) | -4 | - |
| 242 | (29,33) | -2 | - |
| 256 | (29,39) | -5 | - |
| 266 | (29,35) | -3 | - |
| 269 | (30,40) | -5 | + |
| 280 | (31,41) | -5 | - |
| 298 | (32,42) | -5 | + |