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A BAYESIAN THEORY FOR SEISMIC FORESHOCKS AND AFTERSHOCKS

B.-F. APOSTOL

Department of Seismology, Institute of Earth's Physics, Măgurele-Bucharest MG-6, POBox MG-35, Romania, email: apostol@infp.infp.ro

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Abstract. Statistical distributions in time, magnitude and energy are derived for seismic foreshocks and aftershocks accompanying a main seismic shock, as based on the Bayesian theory of probabilities and on a model introduced recently for the accumulation of energy in a seismic focus. Omori's law is obtained as a self-replication of a generating distribution, the self-consistency of the process requiring an exponential law for this generating distribution. The two distributions are inter-related by Euler's transform, which provides also a generalized form of Omori's law. The regime of the accompanying seismic events is characterized as fully as possible, including the time dependence of the magnitude and the rate of released energy.

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Since 1894 when Omori suggested that seismic aftershocks are distributed according to $\sim 1/\tau^{\gamma}$, where $\gamma = 1^{+}$ and τ denotes the time elapsed from the main shock [1], the seismic activity accompanying a major earthquake, both as precursory or subsequent seismic phenomena, has been a matter of debate. One of the major difficulties in advancing knowledge on this subject is the lack of means for distinguishing between seismic events genuinely accompanying a main shock and other, "regular" seisms, superposed over the associated seismic activity, and belonging possibly to other "regular" time series of seismic activity, without any relationship with the main seismic shock. Statistical distributions of such events, both in time, magnitude and energy, may help in operating such a distinction, and it is precisely in this direction that progress has been recorded recently, especially in connnection with the critical point theory for foreshocks and aftershocks, as based on self-organized criticality [2–4].

The aim of this paper is to describe a derivation of Omori's law for the distributions of the acompanying seismic activity in time, magnitude and energy, as based on the concept of statistical ensembles. It is also based on a model discussed recently, regarding the accumulation of seismic energy in a seismic source, and on the probability distributions in time, magnitude and energy of "regular" earthquakes [5]. A brief description of the essentials of this model is given in **Appendix 1**. It is shown that Omori's law arises by a self-replication process of a generating distribution, which is given by an exponential law for the self-consistency of the process. The two distributions are inter-related by Euler's transform, which provides also a generalized form of Omori's law.

To begin with, it is assumed that there may exist an associated seismic activity accompanying a main seismic event, as seismic foreshocks and aftershocks, and this whole "secondary" seismic activity forms a statistical ensemble, *i.e.*, it is described by probability distributions.

Let the main shock occur at a critical time $t_c = 0$, and measure time τ of the accompanying seismic activity with respect to this initial moment of time. Time τ takes both positive values, for aftershocks, and negative values, for foreshocks. As this seismic activity corresponds to pairs of events separated by time τ, then the corresponding statistical distributions are functions of the absolute value $|\tau|$ of time τ, as pointed out in earlier studies [6–8]. It is shown in **Appendix 2** that the associated seismic activity proceeds by the self-replication of a generating distribution of accompanying events, the self-consistency of the process requiring an exponential form for the generating distribution. It amounts to viewing the accompanying seismic activity as a relaxation to equilibrium of the seismic zone, and the corresponding statistical distribution $p(\tau)$ can be obtained formally by using the principle of the maximal entropy $S = -\int d\tau \cdot p(\tau) \ln p(\tau)$. In order to fully characterize this associated seismic activity, a mean value t_c' of its duration may be introduced, where t_c' may be viewed as a characteristic scale time. By standard procedure the temporal probability distribution

$$p(\tau) = \alpha e^{-\alpha|\tau|}, \quad \alpha = 1/t_c' \tag{1}$$

is obtained straightforwardly, as the generating distribution for the seisms accompanying a main shock. In general, the characteristic time t_c' may depend not only on the nature of the seismic source and the seismic zone, but also on the magnitude of the main shock. On the other hand, the distribution of the accompanying events can be obtained directly from (6) in **Appendix 1**, by expanding the temporal probability of the main shocks in powers of $|\tau|$ in the neighbourhood of a main shock with mean recurrence time t_r. It is easy to see that replacing $t = t_r$ by $t = t_r + |\tau|$ in (6), where $|\tau| \ll t_r$, the time distribution $p(\tau) \sim (1 + |\tau|/t_r)^{-2} \sim e^{-2|\tau|/t_r}$ can be extracted from the pair distribution, as corresponding to the accompanying seismic activity. It follows that parameter α in

(1) is given by $\alpha = 2/t_r$, and the characteristic time $t_c' = t_r/2$, where t_r is the mean recurence time of the main shock, as given in **Appendix 1**. For large values of time t_r the distribution of the accompanying events has a long tail, but the corresponding time probability is very low. In contrast, the acompanying seismic activity ends quickly for small main shocks, characterized by a small value of the mean recurrence time t_r.

It is shown in **Appendix 2** that the self-replication process of the generating distribution given by (1) leads to the distribution $P(\tau) = \alpha/(e^{\alpha|\tau|} - 1)$ for the seismic events accompanying a major earthquake, which is Omori's law $P(\tau) = 1/|\tau|$ for $\alpha\tau \ll 1$. It may be extended to $\tau \to \infty$ as $P(\tau) = \tau_c^{\gamma-1}/|\tau|^{\gamma}$, where $\gamma = 1^{+}$ and τ_c is a lower-bound cutoff. This result is valid in general, for any finite generating distribution p, the two distribution p and P being inter-related by Euler's transform. This relationship provides also a generalized Omori's law, which is included in **Appendix 2**. According to Omori's law, the accompanying events are concentrated in the neighbourhood of the lower-bound cutoff τ_c. It might also be noted, according to Omori's law, that number n of associated seismic events goes like $\mathrm{d}n/\mathrm{d}\tau \sim 1/|\tau|$ [4, 9].

Omori's law associated to distribution $p(x)$ can be viewed as "conditioned" by the original probability distribution $p(x)$. In so far as the present procedure contemplates such conditioned probabilities, and assigns probabilities to the frequencies of occurrence, it might be said that it is an illustration of the Bayesian theory of probabilities.

A distribution similar to (1) holds also for the difference in magnitude of the associated seisms with respect to the main shock. Indeed, it is well-known [10–13] that the distribution in magnitude M reads $P(M) = \beta e^{-\beta M}$, where magnitude M is related to the released seismic energy by Gutenberg-Richter relationship $\ln E = a + bM$. Values $a \simeq 10$ and $b \simeq 3.5$ may be adopted for these parameters (as corresponding to moderate and strong earthquakes $5.8 < M < 7.3$, for energy in J) [14], which lead to $\beta \simeq 1.17$ within the model of accumulating seismic energy in a localized seismic source [15]. This value is in fair agreement $(\beta \simeq 1.38)$ with the empirical data analyzed by the recurrence relationship [14, 16] $\ln N_{>} = -\ln t_0 - \beta M$, where $N_{>}$ is the number of earthquakes per unit time with magnitude greater than M. For a main shock of magnitude M_0 the probability distribution can be written as $\sim e^{-\beta m} e^{-\beta M}$, where $m = M_0 - M$ is the difference in magnitude between the main shock and an accompanying seismic event of magnitude M. It is worth noting that this factorization of probabilities corresponds to Bayes' theory of probabilities, and, in this spirit, and assuming that the main shock is included in the associated seisms, negative values must also be allowed for the statistical variable $m = M - M_0$, which leads to $\beta e^{-\beta|m|}$ for the distribution in magnitude difference, as suggested previously [17]. It may also be noted that such a distribution can be obtained by the principle of the maximal entropy as $\beta' e^{-\beta'|m|}$, and, since this probability must be equal to the probability of the main shock at $m = 0$, it follows that $\beta' = \beta$. Another observation might also be that associated seisms do follow the same exponential distribution in magnitudes like the "regular" earthquakes.

It is worth noting that, by making use of the exponential distribution in magnitude difference and the temporal distribution given by (1), the time dependence $|m| = (\alpha/\beta)|\tau|$ is obtained, or $\mathrm{d}m/\mathrm{d}\tau = \alpha/\beta$, or, equivalently, the time dependence $M = M_0 - (\alpha/\beta)|\tau|$ of the magnitude of the accompanying seisms, so that it may be estimated that this associated seismic activity is extinct in time $\tau_0 = \beta M_0/\alpha = \beta M_0 t'_c$. As described above, for small values of m ($|m| < 1/\beta$) the distribution in magnitude difference obeys the same Omori-type law $\sim \mathrm{d}m/|m|$ (the lower bound correponding to $m_c = (\alpha/\beta)\tau_c$). The mean difference in magnitude $\bar{m}$ vanishes for the distribution $\beta e^{-\beta|m|}$ ($\bar{m} = 0$), so it is reasonable to employ the dispersion $\delta m = (m^2)^{1/2} = \sqrt{2}/\beta$ as a measure of the average deviation in magnitudes of the accompanying seismic activity. Such an estimation is also consistent with the assumption that the associated seismic activity represent a relaxation regime of the seismic activity. Making use of $\beta \simeq 1.17$ the value $\delta m \simeq 1.2$ is obtained, which is suggestive for the numerical value indicated by Bath's empirical law [18]. A similar analysis, though on a different conceptual basis, was made recently for the accompanying seismic activity [19, 20]. It is worth noting that variance $\delta m = \sqrt{2}/\beta$ occurs in time $\tau_B = (\beta/\alpha)\delta m = \sqrt{2} t'_c$.

The distribution in energy can be written as [5] (see **Appendix 1**)

$$P(E) = \frac{r/E_0}{(1 + E/E_0)^{1+r}}, \tag{2}$$

where $E_0 = e^a$ is the threshold energy in the Gutenberg-Richter relationship, and the parameter r depends on the nature and geometry of the seismic source. Within the model of accumulating seismic energy in a localized seismic focus the value $r = 1/3$ is adopted for this parameter. For large energies, and making use of the Gutenberg-Richter relationship $E/E_0 = e^{br}$, the magnitude distribution $P(M) = \beta e^{-\beta M}$ is obtained straightforwardly from (2), where $\beta = br$. Similar power-law distributions in energy have also been obtained recently, by using the

Tsalis entropy [21]. The distribution (2) can also be written as $P(E) = = (rE_0^r/E^{1+r})(1+E_0/E)^{-1-r}$, where the factor in the first paranthesis may be assigned to energy E_{max} of the main seismic shock, while the factor in the second paranthesis may be assigned to energy $\varepsilon = E$ corresponding to an accompanying seism. Such an approximation is valid for energy E values close to the energy E_{max}, and accounts for the fact that, according to the model of accumulating seismic energy, equation (2) refers to mean values of energy. This procedure serves also to disentangle the accompanying seismic activity from the main shock, and illustrates again the Bayesian theory of probabilities. In addition, it is consistent with the afore-reached conclusion that the associated seismic activity is governed by the same distribution in magnitudes as the main activity. On the other hand, the resulting decomposition indicates that the statistical variable corresponding to energy for the accompanying seisms is actually $x = 1/\varepsilon$, so that the "energy" distribution for the associated seismic activity can be written down as $p(x) \sim (1+E_0/\varepsilon)^{-1-r} = \exp[-(1+r)\ln(1+E_0/\varepsilon)]$, or, finally,

$$p(x) \simeq E_0(1+r)e^{-(1+r)E_0 x}, \quad x = 1/\varepsilon. \tag{3}$$

It may be noted that this distribution is similar to the exponential distributions in time or magnitude, as corresponding to a statistical ensemble at equlibrium, with a characteristic scale energy $(1+r)E_0$. By comparing (3) and (1) the time dependence $\varepsilon = (1+r)E_0 t_c'/|\tau|$ is obtained straightforwardly for the released energy, or the rate

$$\mathrm{d}\varepsilon/\mathrm{d}|\tau| = -(1+r)E_0 t_c/\tau^2 \tag{4}$$

of the energy released in the accompanying seismic activity. Such an $\sim 1/\tau^2$-law seems to be supported by empirical data [4, 9]. Similarly, the magnitude dependence $\varepsilon = (1+r)E_0/\beta|m|$ of the released energy is obtained, and an Omori-type law $\sim \mathrm{d}x/x \sim -\mathrm{d}\varepsilon/\varepsilon$ for small values of x (large values of released energy ε. It must be emphasized that distribution (3) is only valid for small values of x.

In conclusion, it may be said that Omori's distributions in time, magnitude and energy have been derived for the seismic activity accompanying a main seismic shock. It is shown that Omori's law arises by a mechanism of self-replication of a generating distribution, and the self-consistency of this mechanism requires an exponential law for this generating distribution. The two distributions are inter-related by Euler's transform which provides also a generalized form of Omori's law. Time dependence of the magnitudes and the released energy in the accompanying seismic activity are also given.

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APPENDIX 1

A model of accumulating energy in a localized seismic focus. Distribution of regular earthquakes

The Gutenberg-Richter relationship $\ln E = a + bM$ involves a scale energy $E_0 = e^a$, so that the relationship may also read $\ln(E/E_0) = bM$, or $E/E_0 = e^{bM}$. If a volume $\sim R^3$ is ascribed to energy E, where R is a characteristic radius, then a similar scale radius R_0 should be associated to the threshold energy E_0, leading to $\ln(R/R_0) = bM/3$. These characteristic radii can be associated either to the extent of the region disrupted by the earthquakes' effects, [22, 23] or to the spatial extent of the seismic focus where the seismic energy is accumulating.

If the acumulation of energy in a seismic source is characterized by a velocity **v**, then the process of accumulation of energy in time t is governed by the continuity equation $\partial E/\partial t = -\mathbf{v} gradE$. For a uniform compression with the same velocity v along the three spatial coordinates, and a localized focus, this equation can be written as $\mathrm{d}E/\mathrm{d}t = 3v(E + E_0)/(R + R_0)$. For a different geometry of the source, or a different nature of the accumulation process, the factor 3 in this equation may change, so, in the interest of preserving generality, it is replaced by $1/r$. For the present discussion $r = 1/3$, it being derived from purely geometric reasons. Since $R/v = t$, a threshold, or cutoff, time $t_0 = R_0/v$ is also introduced, such that the afore equation reads $r\mathrm{d}E/\mathrm{d}t = (E + E_0)/(t + t_0)$, whose solution is

$$1 + t/t_0 = (1 + E/E_0)^r . \qquad (5)$$

This is the equation of accumulating seismic energy in a localized focus employed in the main text.

Let N_0 be the number of earthquakes that may appear in a long time T with a mean succession time $t_0 = T/N_0$. Similarly, $N = T/(t_0 + t)$ is the number of earthquakes with a mean recurrence time t. Parameter $1/t_0$ can be viewed as the seismicity rate, and the frequency can be written as [24] $N/N_0 = 1/(1 + t/t_0)$, so the temporal probability reads

$$-\mathrm{d}N/N_0 = \frac{1/t_0}{(1 + t/t_0)^2}\mathrm{d}t \qquad (6)$$

It is now easy to get the distribution in energy

$$P(E) = \frac{r/E_0}{(1 + E/E_0)^{1+r}} , \qquad (7)$$

by making use of (5), and the distribution in magnitudes $P(M)=\beta e^{-\beta M}$ for large energies $(E \gg E_0)$, where $\beta = br$, employed in the main text.

Making use of these distributions the Gutenberg-Richter "ocurrence" relationship $\ln(\Delta N/T)=\ln(\beta\Delta M/t_0)-\beta M$ is obtained straightforwardly for the rate $\Delta N/T$ of earthquakes with magnitude in the range M to $M+\Delta M$, and the recurence relationship $\ln(N_>/T)=-\ln t_0-\beta M$ for the rate of $N_>$ earthquakes with magnitude greater than M. Time in (5) may be viewed as the mean recurrence time $t_r = t_0 e^{\beta M}$, or, for earthquakes with magnitude ranging from M to $M+\Delta M$, the recurence time is $t_r = (t_0/\beta\Delta M)e^{\beta M}$. The temporal distribution of such regular, or main shocks, with a fixed mean reccurence time can be obtained easily as $(1/t_r)e^{-t/t_r}$, which implies a deviation $\delta t_r = (\overline{t^2})^{1/2} - t_r = (\sqrt{2}-1)t_r$.

APPENDIX 2

Generalized Omori's law and Euler's transform

Let $p(x)=\mathrm{d}n/\mathrm{d}x$ be a finite distribution over the range $x>0$. The number $\mathrm{d}n_0 = p_0\mathrm{d}x$ of events placed at origin, where $p(0)=p_0$, can be viewed as the number of main events, while the rest of events, distributed over $x>0$, can be viewed as being produced by the main events at a rate $r(x)$, such that

$$p(x)=p_0 r(x). \tag{8}$$

Since the events are not differentiated otherwise except by their position x, it follows that distribution p is also produced at $x+y$ by its value at x multiplied by rate $r(y)$, *i.e.*

$$p(x+y)=p(x)r(y), \tag{9}$$

for any $x, y>0$. This is a self-generating distribution, and equation (9) expresses a self-consistency character of distribution $p(x)$. It may also be written as $p(x+\Delta x)=r(\Delta x)p(x)$, which leads to $\mathrm{d}p/\mathrm{d}x=(-p_1/p_0)p(x)$, where $-p_1 = p'(0)<0$ is the first derivative of $p(x)$ at origin. It follows immediately, from (8) and (9), that distribution $p(x)$ is given by an exponential law $p(x)=p_0 e^{-p_1 x/p_0}$, which can be transformed into a normalized probability distribution $p(x)=p_0 e^{-p_0 x}$. It may be viewed as the probability of occurence of accompanying events, or the generating probability for such events.

The self-generation process implies also the self-replication of the events, such that the distribution $P(x)$, giving the total number of events $P(x)\mathrm{d}x$ in the range x to $x+dx$, obeys the relationship

$$P(x)=p(x)+r(x)P(x)=p(x)+\frac{p(x)}{p_0}P(x). \tag{10}$$

It follows that distribution $P(x)$ is given by

$$P(x)=\frac{p(x)}{1-p(x)/p_0}, \tag{11}$$

which is Euler's transform between $p(x)/p_0$ and $-P(x)/p_0$. Distribution $P(x)$ as given by (11) corresponds to all the events generated in the process of producing accompanying events by the main events placed at $x=0$. It is worth noting that $P(x)$ is singular at origin. Introducing the exponential distribution $p(x)=p_0e^{-p_0x}$ in (11) we get

$$P(x)=\frac{p_0}{e^{p_0x}-1}, \tag{12}$$

which is Omori's law $P(x)=1/x$ for $p_0x\ll 1$. It is customary to introduce a lower-bound cutoff x_c and to extend $1/x$ to infinite as $x_c^{\gamma-1}/x^{\gamma}$, where $\gamma=1^+$, such that

$$\int_{x_c}^{\infty}\mathrm{d}x\frac{p_0}{e^{p_0x}-1}=\int_{x_c}^{\infty}\mathrm{d}x(x_c^{\gamma-1}/x^{\gamma}). \tag{13}$$

Equation (13) gives the exponent $\gamma=1-1/\ln(p_0x_c)=1^+$ in the limit $x_c\to 0$.

It might be noted that $P(x)$ as given by (12) is, formally, a Bose-Einstein-type occupation number (in two dimensions) for an exponential, Boltzmann-type, distribution $p(x)$. The self-replication equation (10), which describes a geometric series, has also a formal resemblance to Dyson's equation in the theory of interacting many-body ensembles. Distributions $P(x)$ as given by Euler's transform (11) can be considered for a general form of generating distributions $p(x)$, which amounts to including only the self-replication process for the accompanying events produced by $p(x)=p_0r(x)$. For this general case, the series expansion $p(x)=p_0-p_1x...$ can be considered in the neighbourhood of $x=0$, leading to Omori's law $P(x)=p_0x_0/x$ for $x\ll x_0=p_0/p_1$. Euler's transform (11) provides a general representation $P(x)=p_0/h(x)$ for such singular distributions, where $h(0)=0$ and $h(\infty)\to\infty$ (such that, preferrably, $P(x)$ is integrable at infinite). It implies $p(x)=p_0(1-h)\simeq p_0/(1+h)$ for $x\to 0$. Such a representation may be regarded as a generalized Omori-type distribution. Equation (12) gives, actually, such a generalized Omori's law. For $h(x) \sim x^{\gamma}$, $\gamma > 0$, power-law distributions $P(x) \sim 1/x^{\gamma}$ are obtained (an upper-bound cutoff D is necessary for $0 < \gamma \leq 1$, as well as a lower-bound cutoff x_c for $1 \leq \gamma$).

In order to derive equation (3) in the main text we may use the pair probability $\Pi(\Delta E)$ defined as

$$\Pi(\Delta E) = P(E + \Delta E)/P(E),$$

where $P(E)$ is given by equation (2) in the main text. We get straightforwardly

$$\Pi(\Delta E) = e^{-(1+r)\Delta E?(E_0+\Delta E)}.$$

This equation may be viewed as giving the probability of occurrence of a pair of foreshock-main shock, or main shock-aftershock. Since there is no other special distinction between the members of such a pair, and any pair of earthquakes may be given such an interpretation from the statistical standpoint, we may take $\Delta E = E_0$, and for $E \gg E_0$ we get thereby equation (3) in the main text.

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