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QUANTUM FIDELITY FOR THE STATES OF THE DAMPED HARMONIC OSCILLATOR

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Abstract. Using the expression of the fidelity for the most general Gaussian quantum states, the behaviour of the quantum fidelity is described for the states of a harmonic oscillator interacting with an environment, in particular with a thermal bath. By taking a correlated squeezed Gaussian state as initial state, we calculate the quantum fidelity for both kinds of undisplaced and displaced states, and for different values of the squeezing and correlation parameters and of the environment temperature.

Key words: quantum information theory, quantum fidelity, open systems.

1. INTRODUCTION

In quantum information theory it is very important to quantify the similarity or distinguishability of quantum states. There are several distance measures widely used to distinguish quantum states in quantum information transfer, in particular the trace distance and quantum fidelity [1]. Using Uhlmann's transition probability [2], Jozsa [3] defined the quantum fidelity of two quantum states. The fidelity quantifies how close are the two states, and their separation can be measured, for example, by the Bures distance [4].

In the recent years there has been an increased interest in the study of Gaussian states of continuous variable systems used in quantum information processing [5]. Usually the quantum fidelity for quantum optics experiments, in particular quantum teleportation experiments, is calculated for pure coherent states as input states [6, 7, 8]. However, in real experiments the input quantum states have some non-negligible degree of mixedness, achieved

mainly due to the decoherence phenomenon which takes place during the interaction of the system with its environment. In the framework of the theory of open systems based on quantum dynamical semigroups, we consider a system consisting of an one-dimensional harmonic oscillator interacting with an environment, in particular with a thermal bath. Our purpose is to study the influence of the environment on the time evolution of the quantum fidelity for Gaussian states of the considered system.

The structure of the paper is the following. In Section 2 we briefly review the basic formalism for the calculation of the quantum fidelity for a general pair of single-mode Gaussian states. The time evolution of the harmonic oscillator in the theory of open quantum systems is described in Section 3. By taking a correlated squeezed Gaussian state as initial state, the quantum fidelity of the initial state and an arbitrary time state of the considered system is calculated in Section 4. A summary is given in Section 5.

2. FIDELITY OF GAUSSIAN STATES

For two quantum states, described by the density operators ρ_1 and ρ_2 , the fidelity F is defined as the Uhlmann's transition probability by [2]

$$F(\rho_1, \rho_2) = [\text{Tr}(\sqrt{\rho_1} \rho_2 \sqrt{\rho_1})^{1/2}]^2. \quad (1)$$

The fidelity F is continuous with respect to ρ_1 and ρ_2 and it has the following natural properties as a measure of the closeness of the two states:

- (i) $0 \leq F(\rho_1, \rho_2) \leq 1$ and $F(\rho_1, \rho_2) = 1$ if and only if $\rho_1 = \rho_2$;
- (ii) $F(\rho_1, \rho_2) = F(\rho_2, \rho_1)$ (symmetry);
- (iii) $F(\rho_1, \rho_2) = \langle \psi_1 | \rho_2 | \psi_1 \rangle$ if one of the states is pure — i.e., when $\rho_1 = |\psi_1\rangle\langle\psi_1|$;
- (iv) $F(\rho_1, \rho_2)$ is invariant under a unitary transformation U on the state space — i.e., $F(U\rho_1U^\dagger, U\rho_2U^\dagger) = F(\rho_1, \rho_2)$.

Gaussian states have positive definite Wigner functions which may be interpreted as probability distributions in phase space. A general single-mode Gaussian state ρ is completely characterized by its first and the second moments and can be represented by a correlated squeezed state [9] or by a squeezed and displaced thermal state [10].

The fidelity of two undisplaced thermal states was calculated in Ref. [11] and that of displaced thermal states in Ref. [12]. The fidelity of a large class of single-mode Gaussian states was obtained in Ref. [13].

The technical difficulty in calculating the fidelity comes from the square root of operators. For Gaussian states the difficulty is solved, since the characteristic function of the square root of a Gaussian state is also Gaussian. By

a successive use of the composition rule in the position representation,

$$\langle x|\rho_1\rho_2|y\rangle = \int dz \langle x|\rho_1|z\rangle \langle z|\rho_2|y\rangle, \quad (2)$$

with

$$\langle x|\rho|y\rangle = \exp[-(ax^2 + dy^2 + 2bxy) + lx + ky + g], \quad (3)$$

it was obtained in Ref. [13] the quantum fidelity of a thermal (undisplaced and unsqueezed) state ρ_1 and a displaced squeezed state ρ_2 . Denoting the amplitude of ρ_2 by the row vector $\alpha \equiv \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix}$ and introducing the matrices \mathbf{A}_i of the form

$$\mathbf{A} = \begin{pmatrix} a_{qq} & a_{pq} \\ a_{pq} & a_{pp} \end{pmatrix}, \quad (4)$$

whose elements are connected with the variances σ_{qq} , σ_{pp} and covariance σ_{pq} of canonical position q and momentum p operators, respectively, through the relations $a_{qq} = 2\sigma_{qq}$, $a_{pp} = 2\sigma_{pp}/\hbar^2$, $a_{pq} = 2\sigma_{pq}/\hbar$, we have [13]

$$F = \frac{2}{\sqrt{\Delta + \delta} - \sqrt{\delta}} \exp[-\alpha^T (\mathbf{A}_1 + \mathbf{A}_2)^{-1} \alpha], \quad (5)$$

where

$$\Delta = \det(\mathbf{A}_1 + \mathbf{A}_2), \quad \delta = (\det \mathbf{A}_1 - 1)(\det \mathbf{A}_2 - 1). \quad (6)$$

Using the property (iv) and denoting the mean amplitudes of ρ_1 and respectively ρ_2 by $\alpha_i \equiv \begin{pmatrix} \alpha_{ix} \\ \alpha_{iy} \end{pmatrix}$ ($i = 1, 2$), we obtain a similar formula for the quantum fidelity of two general Gaussian quantum states:

$$F = \frac{2}{\sqrt{\Delta + \delta} - \sqrt{\delta}} \exp[-\beta^T (\mathbf{A}_1 + \mathbf{A}_2)^{-1} \beta], \quad (7)$$

where now $\beta = \alpha_2 - \alpha_1$.

If ρ_1 is a pure state then $\det \mathbf{A}_1 = 1$ and the fidelity becomes

$$F = \frac{1}{\sqrt{\det((\mathbf{A}_1 + \mathbf{A}_2)/2)}} \exp[-\beta^T (\mathbf{A}_1 + \mathbf{A}_2)^{-1} \beta]. \quad (8)$$

3. MASTER EQUATION FOR THE HARMONIC OSCILLATOR

In the axiomatic formalism of introducing dissipation in quantum mechanics, based on quantum dynamical semigroups, the irreversible time evolution of the open system is described by the following general quantum Markovian master equation for the density operator $\rho(t)$ [14]:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar} \sum_j ([V_j \rho(t), V_j^\dagger] + [V_j, \rho(t) V_j^\dagger]). \quad (9)$$

Here H is the Hamiltonian operator of the system and V_j, V_j^\dagger are operators on the Hilbert space of the Hamiltonian, which model the environment. In the case of an exactly solvable model for the damped harmonic oscillator, the two possible operators V_1 and V_2 are taken as linear polynomials in coordinate q and momentum p [15, 16] and the harmonic oscillator Hamiltonian H is chosen of the most general quadratic form

$$H = H_0 + \frac{\mu}{2}(qp + pq), \quad H_0 = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2. \quad (10)$$

With these choices the master equation (9) takes the following form:

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i}{\hbar}[H_0, \rho] - \frac{i}{2\hbar}(\lambda + \mu)[q, \rho p + p\rho] + \frac{i}{2\hbar}(\lambda - \mu)[p, \rho q + q\rho] \\ & - \frac{D_{pp}}{\hbar^2}[q, [q, \rho]] - \frac{D_{qq}}{\hbar^2}[p, [p, \rho]] + \frac{D_{pq}}{\hbar^2}([q, [p, \rho]] + [p, [q, \rho]]). \end{aligned} \quad (11)$$

The quantum diffusion coefficients D_{pp}, D_{qq}, D_{pq} and the dissipation constant λ satisfy the following fundamental constraints: $D_{pp} > 0, D_{qq} > 0$ and

$$D_{pp}D_{qq} - D_{pq}^2 \geq \frac{\lambda^2 \hbar^2}{4}. \quad (12)$$

In the particular case when the asymptotic state is a Gibbs state $\rho_G(\infty) = e^{-\frac{H_0}{kT}} / \text{Tr} e^{-\frac{H_0}{kT}}$, these coefficients have the form [15, 16]

$$D_{pp} = \frac{\lambda + \mu}{2} \hbar m \omega \coth \frac{\hbar \omega}{2kT}, \quad D_{qq} = \frac{\lambda - \mu}{2} \frac{\hbar}{m \omega} \coth \frac{\hbar \omega}{2kT}, \quad D_{pq} = 0, \quad (13)$$

where T is the temperature of the thermal bath. In this case, the fundamental constraints are satisfied only if $\lambda > \mu$ and

$$(\lambda^2 - \mu^2) \coth^2 \frac{\hbar \omega}{2kT} \geq \lambda^2. \quad (14)$$

The necessary and sufficient condition for translational invariance is $\lambda = \mu$ [15, 16]. In the following general values $\lambda \neq \mu$ will be considered. In this

way we violate translational invariance, but we keep the canonical thermal equilibrium state.

From the master equation (11) we obtain the following equations of motion for the expectation values σ_q and σ_p of coordinate and momentum:

$$\frac{d\sigma_q(t)}{dt} = -(\lambda - \mu)\sigma_q(t) + \frac{1}{m}\sigma_p(t), \quad (15)$$

$$\frac{d\sigma_p(t)}{dt} = -m\omega^2\sigma_q(t) - (\lambda + \mu)\sigma_p(t). \quad (16)$$

In the underdamped case ($\omega > \mu$) considered in this paper, with the notation $\Omega^2 \equiv \omega^2 - \mu^2$, we obtain [15, 16]:

$$\sigma_q(t) = e^{-\lambda t} \left((\cos \Omega t + \frac{\mu}{\Omega} \sin \Omega t)\sigma_q(0) + \frac{1}{m\Omega} \sin \Omega t \sigma_p(0) \right), \quad (17)$$

$$\sigma_p(t) = e^{-\lambda t} \left(-\frac{m\omega^2}{\Omega} \sin \Omega t \sigma_q(0) + (\cos \Omega t - \frac{\mu}{\Omega} \sin \Omega t)\sigma_p(0) \right) \quad (18)$$

and $\sigma_q(\infty) = \sigma_p(\infty) = 0$.

From the master equation (11) we can also obtain the equations of motion for the variances σ_{qq} , σ_{pp} and covariance σ_{pq} of coordinate and momentum which are needed to calculate the uncertainty function:

$$\frac{d\sigma_{qq}(t)}{dt} = -2(\lambda - \mu)\sigma_{qq}(t) + \frac{2}{m}\sigma_{pq}(t) + 2D_{qq}, \quad (19)$$

$$\frac{d\sigma_{pp}(t)}{dt} = -2(\lambda + \mu)\sigma_{pp}(t) - 2m\omega^2\sigma_{pq}(t) + 2D_{pp}, \quad (20)$$

$$\frac{d\sigma_{pq}(t)}{dt} = -m\omega^2\sigma_{qq}(t) + \frac{1}{m}\sigma_{pp}(t) - 2\lambda\sigma_{pq}(t) + 2D_{pq}. \quad (21)$$

Introducing the notations

$$X(t) = \begin{pmatrix} m\omega\sigma_{qq}(t) \\ \sigma_{pp}(t)/m\omega \\ \sigma_{pq}(t) \end{pmatrix}, \quad D = \begin{pmatrix} 2m\omega D_{qq} \\ 2D_{pp}/m\omega \\ 2D_{pq} \end{pmatrix}, \quad (22)$$

the solutions of these equations of motion can be written in the form [15, 16]

$$X(t) = (Te^{Kt}T)(X(0) - X(\infty)) + X(\infty), \quad (23)$$

where the matrices T and K are given by

$$T = \frac{1}{2i\Omega} \begin{pmatrix} \mu + i\Omega & \mu - i\Omega & 2\omega \\ \mu - i\Omega & \mu + i\Omega & 2\omega \\ -\omega & -\omega & -2\mu \end{pmatrix},$$

$$K = \begin{pmatrix} -2(\lambda - i\Omega) & 0 & 0 \\ 0 & -2(\lambda + i\Omega) & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \quad (24)$$

and

$$X(\infty) = -(TK^{-1}T)D. \quad (25)$$

Formula (25) gives a simple connection between the asymptotic values ($t \rightarrow \infty$) of $\sigma_{qq}(t)$, $\sigma_{pp}(t)$, $\sigma_{pq}(t)$ and the diffusion coefficients D_{pp} , D_{qq} , D_{pq} :

$$\sigma_{qq}(\infty) = \frac{(m\omega)^2(2\lambda(\lambda + \mu) + \omega^2)D_{qq} + \omega^2 D_{pp} + 2m\omega^2(\lambda + \mu)D_{pq}}{2(m\omega)^2\lambda(\lambda^2 + \omega^2 - \mu^2)}, \quad (26)$$

$$\sigma_{pp}(\infty) = \frac{(m\omega^2)^2 D_{qq} + (2\lambda(\lambda - \mu) + \omega^2)D_{pp} - 2m\omega^2(\lambda - \mu)D_{pq}}{2\lambda(\lambda^2 + \omega^2 - \mu^2)}, \quad (27)$$

$$\sigma_{pq}(\infty) = \frac{-(\lambda + \mu)(m\omega)^2 D_{qq} + (\lambda - \mu)D_{pp} + 2m(\lambda^2 - \mu^2)D_{pq}}{2m\lambda(\lambda^2 + \omega^2 - \mu^2)}. \quad (28)$$

These asymptotic values do not depend on the initial values $\sigma_{qq}(0)$, $\sigma_{pp}(0)$, $\sigma_{pq}(0)$ and in the case of a thermal bath with coefficients (13), they reduce to [15, 16]

$$\sigma_{qq}(\infty) = \frac{\hbar}{2m\omega} \coth \frac{\hbar\omega}{2kT}, \quad \sigma_{pp}(\infty) = \frac{\hbar m\omega}{2} \coth \frac{\hbar\omega}{2kT}, \quad \sigma_{pq}(\infty) = 0. \quad (29)$$

In the considered underdamped case we have

$$Te^{Kt}T = -\frac{e^{-2\lambda t}}{2\Omega^2} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad (30)$$

where b_{ij} , $i, j = 1, 2, 3$ are time-dependent oscillating functions given by (3.78) in [15].

The relation (12) is a necessary condition for the generalized uncertainty inequality

$$\sigma(t) \equiv \sigma_{qq}(t)\sigma_{pp}(t) - \sigma_{pq}^2(t) \geq \frac{\hbar^2}{4} \quad (31)$$

to be fulfilled, where $\sigma(t)$ is the Schrödinger generalized uncertainty function. The equality in relation (31) is realized for a special class of pure states, called correlated coherent states [9] or squeezed coherent states.

4. CALCULATION OF QUANTUM FIDELITY

We consider a harmonic oscillator with an initial Gaussian wave function

$$\Psi(q) = \left(\frac{1}{2\pi\sigma_{qq}(0)} \right)^{\frac{1}{4}} \exp \left[-\frac{1}{4\sigma_{qq}(0)} \left(1 - \frac{2i}{\hbar} \sigma_{pq}(0) \right) (q - \sigma_q(0))^2 + \frac{i}{\hbar} \sigma_p(0) q \right], \quad (32)$$

where $\sigma_{qq}(0)$ is the initial spread, $\sigma_{pq}(0)$ the initial covariance, and $\sigma_q(0)$ and $\sigma_p(0)$ are the initial averaged position and momentum of the wave packet. The initial state (32) represents a correlated coherent state [9] with the variances and covariance of coordinate and momentum

$$\sigma_{qq}(0) = \frac{\hbar\delta}{2m\omega}, \quad \sigma_{pp}(0) = \frac{\hbar m\omega}{2\delta(1-r^2)}, \quad \sigma_{pq}(0) = \frac{\hbar r}{2\sqrt{1-r^2}}. \quad (33)$$

Here, δ is the squeezing parameter which measures the spread in the initial Gaussian packet and r , with $|r| < 1$ is the correlation coefficient at time $t = 0$. The initial values (33) correspond to a minimum uncertainty state, since they fulfil the generalized uncertainty relation

$$\sigma_{qq}(0)\sigma_{pp}(0) - \sigma_{pq}^2(0) = \frac{\hbar^2}{4}. \quad (34)$$

For $\delta = 1$ and $r = 0$ the correlated coherent state becomes a Glauber coherent state.

The finite temperature Schrödinger generalized uncertainty function $\sigma(t)$, calculated in Ref. [17], has the expression

$$\begin{aligned} \sigma(t) = & \frac{\hbar^2}{4} \left\{ e^{-4\lambda t} \left[1 - \left(\delta + \frac{1}{\delta(1-r^2)} \right) \coth \epsilon + \coth^2 \epsilon \right] + \right. \\ & + e^{-2\lambda t} \coth \epsilon \left[\left(\delta + \frac{1}{\delta(1-r^2)} - 2 \coth \epsilon \right) \frac{\omega^2 - \mu^2 \cos(2\Omega t)}{\Omega^2} + \right. \\ & \left. \left. + \left(\delta - \frac{1}{\delta(1-r^2)} \right) \frac{\mu \sin(2\Omega t)}{\Omega} + \frac{2r\mu\omega(1 - \cos(2\Omega t))}{\Omega^2 \sqrt{1-r^2}} \right] + \coth^2 \epsilon \right\}, \quad (35) \end{aligned}$$

where we have introduced the notation

$$\epsilon \equiv \frac{\hbar\omega}{2kT}. \quad (36)$$

We are interested to study the evolution of quantum fidelity when the initial state is pure and the system is in interaction with an environment. If the initial wave function is Gaussian, then the density matrix remains Gaussian for all times (with time-dependent parameters which determine its amplitude and spread) and centered along the trajectory, which is given by the solutions $\sigma_q(t)$ and $\sigma_p(t)$ of the dissipative equations of motion [18]. We take the initial Gaussian wave function (32) as pure state 1 with the corresponding matrix \mathbf{A}_1 and the state at an arbitrary time t described by the density matrix $\rho(t)$

as state 2 with the corresponding matrix \mathbf{A}_2 . Then we can apply the formula for the quantum fidelity (8), where the matrices \mathbf{A}_i (4) are given by

$$\mathbf{A}_1 = \begin{pmatrix} a_{qq}(0) & a_{pq}(0) \\ a_{pq}(0) & a_{pp}(0) \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} a_{qq}(t) & a_{pq}(t) \\ a_{pq}(t) & a_{pp}(t) \end{pmatrix}. \quad (37)$$

We obtain

$$\mathbf{A}_1 + \mathbf{A}_2 = 2 \begin{pmatrix} \sigma_{qq}(0) + \sigma_{qq}(t) & \frac{1}{\hbar} (\sigma_{pq}(0) + \sigma_{pq}(t)) \\ \frac{1}{\hbar} (\sigma_{pq}(0) + \sigma_{pq}(t)) & \frac{1}{\hbar^2} (\sigma_{pp}(0) + \sigma_{pp}(t)) \end{pmatrix} \quad (38)$$

and

$$\det \frac{\mathbf{A}_1 + \mathbf{A}_2}{2} = \frac{1}{\hbar^2} [\sigma(0) + \sigma(t) + \sigma_{qq}(0)\sigma_{pp}(t) + \sigma_{pp}(0)\sigma_{qq}(t) - 2\sigma_{pq}(0)\sigma_{pq}(t)]. \quad (39)$$

In the case when the initial Gaussian wave function (32) is not displaced, then $\alpha_1 = \alpha_2 = 0$ and the expression (8) simplifies:

$$F = \frac{1}{\sqrt{\det((\mathbf{A}_1 + \mathbf{A}_2)/2)}}. \quad (40)$$

Then we obtain

$$F(t) = \frac{\hbar}{\sqrt{\sigma(0) + \sigma(t) + \sigma_{qq}(0)\sigma_{pp}(t) + \sigma_{pp}(0)\sigma_{qq}(t) - 2\sigma_{pq}(0)\sigma_{pq}(t)}}. \quad (41)$$

Evidently the fidelity at the initial moment is $F(0) = 1$ and for large times we get:

$$F(t \rightarrow \infty) = \frac{2}{\sqrt{1 + \left(\delta + \frac{1}{\delta(1-r^2)} \right) \coth \epsilon + \coth^2 \epsilon}}. \quad (42)$$

If $r = 0$, then expression (42) becomes:

$$F(\infty) = \frac{2}{\sqrt{1 + \left(\delta + \frac{1}{\delta} \right) \coth \epsilon + \coth^2 \epsilon}} \quad (43)$$

and for a coherent state ($\delta = 1$) we obtain:

$$F(\infty) = \frac{2}{1 + \coth \epsilon}. \quad (44)$$

When the temperature of the environment is $T = 0$, expression (43) becomes:

$$F(\infty) = \frac{2\sqrt{\delta}}{\delta + 1}, \quad (45)$$

and respectively $F(\infty) = 1$, if the initial state is a coherent state. For $\delta \neq 1$, we get always $F(\infty) < 1$.

We see that the asymptotic value of the quantum fidelity depends on the environment temperature and parameters of the initial state. This value is decreasing with increasing temperature and squeezing parameter.

In the case when the initial Gaussian wave function (32) is displaced, then in the previous expressions the exponential factor $\exp(-E)$ of Eq. (8) is present, where E for large times ($t \rightarrow \infty$) has the form

$$E(\infty) = \frac{m\omega\sigma_q^2(0) \left(\frac{1}{\delta(1-r^2)} + \coth \epsilon \right) - 2\sigma_q(0)\sigma_p(0) \frac{r}{\sqrt{1-r^2}} + \sigma_p^2(0) \frac{\delta + \coth \epsilon}{m\omega}}{\hbar \left[1 + \left(\delta + \frac{1}{\delta(1-r^2)} \right) \coth \epsilon + \coth^2 \epsilon \right]}. \quad (46)$$

If $r = 0$, then the exponential factor becomes

$$E(\infty) = \frac{m^2\omega^2\sigma_q^2(0) \left(\frac{1}{\delta} + \coth \epsilon \right) + \sigma_p^2(0)(\delta + \coth \epsilon)}{\hbar m\omega \left[1 + \left(\delta + \frac{1}{\delta} \right) \coth \epsilon + \coth^2 \epsilon \right]} \quad (47)$$

and for a coherent state we obtain:

$$E(\infty) = \frac{m^2\omega^2\sigma_q^2(0) + \sigma_p^2(0)}{\hbar m\omega(1 + \coth \epsilon)}. \quad (48)$$

If, in addition, the temperature of the environment is $T = 0$, then the fidelity is given just by the exponential factor:

$$F(\infty) = \exp \left(-\frac{m^2\omega^2\sigma_q^2(0) + \sigma_p^2(0)}{2\hbar m\omega} \right). \quad (49)$$

5. SUMMARY

Using the expression of the fidelity for the most general Gaussian quantum states, we have investigated the behaviour of the quantum fidelity for the states of a harmonic oscillator interacting with an environment, in particular with a thermal bath. The time evolution of the considered system was described in the framework of the theory of open systems based on quantum dynamical semigroups. By taking a correlated squeezed Gaussian state as initial state, we calculated the quantum fidelity for both kinds of undisplaced

and displaced states, and for different values of the squeezing and correlation parameters and of the environment temperature.

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