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*Dedicated to Prof. Dumitru Barbu Ion's 70th Anniversary*

# COHERENT STATES ASSOCIATED TO THE JACOBI GROUP

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*Abstract.* We review and discuss some properties of the coherent states associated to the Jacobi group.

*Key words:* coherent states, squeezed states, Jacobi group.

## 1. INTRODUCTION

The coherent states (CS) offer a useful connection between classical and quantum mechanics [1–5]. In several references [6, 7] we have constructed CS attached to the Jacobi group. It is well known that the Jacobi group appears in Quantum Mechanics, Geometric Quantization, Optics [8–16]. The mathematicians have given the name "Jacobi group" to the semidirect product of the Heisenberg-Weyl group and the symplectic group [17]. The same group is known to physicists under other names, as the Schrödinger group [18]. Also the name "Weyl-symplectic" group is used for the same semi-direct product of the Heisenberg-Weyl group and the symplectic group [8, 19, 20].

In [6] we have constructed generalized CS attached to the Jacobi group $G_1^J := H_1 \rtimes \mathrm{SU}(1,1)$, based on the homogeneous Kähler manifold $\mathcal{D}_1^J = H_1/\mathbb{R} \times \times \mathrm{SU}(1,1)/\mathrm{U}(1) = \mathbb{C}^1 \times \mathcal{D}_1$. $\mathcal{D}_n$ denotes the Siegel unit ball [21, 22] and $H_n$ is the $(2n+1)$-dimensional real Heisenberg-Weyl group with Lie algebra $\mathfrak{h}_n$. When expressed in appropriate coordinates on the manifold $\mathfrak{X}_1^J := \mathbb{C} \times \mathcal{H}_1$, where $\mathcal{H}_1$ is the Siegel upper half plane $\mathcal{H}_1 = \{v \in \mathbb{C} \mid \Im(v) > 0\}$, the Kähler two-form $\omega_1$ derived from the Kähler potential obtained from the scalar product of Perelomov's coherent state vectors based on $\mathcal{D}_1^J$, is identical with the one considered by Kähler-Berndt [23–27], here denoted $\omega_1'$. This $\omega_1'$ describes the geometry of manifolds on which the “gaussons” [28] are constructed [7, 3]. In the present paper we also give more details about this identification. We also recall our generalization of the full construction to the “$n$” dimensional Jacobi group $G_n^J = H_n \rtimes \mathrm{Sp}(n, \mathbb{R})$ in [7], where the associated coherent states are based on the manifold $\mathcal{D}_n^J = \mathbb{C}^n \times \mathcal{D}_n$. Our method of finding representations of the Lie algebra is based on our previous papers [29–31], as a particular case of realization of the so called coherent state algebras [22, 32, 33] by holomorphic differential operators with polynomial coefficients.

The paper is laid out as follows. §2 fixes the definition of the Jacobi algebra $\mathfrak{g}_1^J$. §3 recall the construction of CS vectors attached to the Jacobi group $G_1^J$ defined on the manifold $\mathcal{D}_1^J$ and also of the normalized squeezed CS vectors [34]. The main information is extracted from [6], but we also present in Proposition 1 a very suggestive formula of the series expansion for CS vectors. The action of the Jacobi group $G_1^J$ on the base manifold $\mathcal{D}_1^J$ can be extracted from the first part of Proposition 2 of §4, which gives the resolution of unity, sometimes [3] considered as characterization of CS. The Kähler two form $\omega_1$ on $\mathcal{D}_1^J$ is reproduced also in §4. The last section is devoted to Kähler-Berndt’s approach. In Remark 1 we present in full details a proof of a known result from the papers of Kähler and Berndt, the action of the real Jacobi group $G_0^J(\mathbb{R})$ on $\mathfrak{X}_1^J$. During the presentation, I indicate the significance in Physics of the mathematical formulas.

## 2. THE JACOBI ALGEBRA $\mathfrak{g}_1^J$

The Heisenberg-Weyl group is the group with the 3-dimensional real Lie algebra

$$\mathfrak{h}_1 \equiv \mathfrak{g}_{HW} = \left\langle is1 + xa^+ - \bar{x}a \right\rangle_{s\in\mathbb{R},\, x\in\mathbb{C}}, \tag{1}$$

where $a^+$ ($a$) are the boson creation (respectively, annihilation) operators which verify the CCR (4a).

Let us also consider the Lie algebra of the group SU(1, 1):

$$\mathfrak{su}(1,1) = \left\langle 2i\theta K_0 + yK_+ - \bar{y}K_- \right\rangle_{\theta\in\mathbb{R},\, y\in\mathbb{C}}, \tag{2}$$

where the generators $K_{0,+,-}$ verify the standard commutation relations (4b).

The Jacobi algebra is defined as the the semi-direct sum

$$\mathfrak{g}_1^J := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1), \tag{3}$$

where $\mathfrak{h}_1$ is an ideal in $\mathfrak{g}_1^J$, *i.e.* $[\mathfrak{h}_1, \mathfrak{g}_1^J] = \mathfrak{h}_1$, determined by the commutation relations:

$$[a, a^+] = 1, \tag{4a}$$

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0, \tag{4b}$$

$$[a, K_+] = a^+, \quad [K_-, a^+] = a, \tag{4c}$$

$$[K_+, a^+] = [K_-, a] = 0, \tag{4d}$$

$$[K_0, a^+] = \frac{1}{2}a^+, \quad [K_0, a] = -\frac{1}{2}a. \tag{4e}$$

The general scheme [31] associates to elements of the Lie algebra $\mathfrak{g}$ differential operators: $X \in \mathfrak{g} \to \mathbb{X}$. We recall the following result established in [6]:

**Lemma 1.** *The differential action of the generators* (4a)–(4e) *of the Jacobi algebra* (3) *is given by the formulas:*

$$a = \frac{\partial}{\partial z}; \quad a^+ = z + w\frac{\partial}{\partial z}; \tag{5a}$$

$$\mathbb{K}_- = \frac{\partial}{\partial w}; \quad \mathbb{K}_0 = k + \frac{1}{2}z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}; \tag{5b}$$

$$\mathbb{K}_+ = \frac{1}{2}z^2 + 2kw + zw\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w}, \tag{5c}$$

*where* $z \in \mathbb{C}$, $|w| < 1$.

In the proof of Lemma 1, the (un-normalized) Perelomov's CS vectors are used, and this construction is recalled in the next section.

## 3. COHERENT STATE VECTORS

Let us suppose that we know the derived representation $d\pi$ of the Lie algebra $\mathfrak{g}_1^J$ (3) of the Jacobi group $G_1^J$. We associate to the generators $a$, $a^+$ of the HW-group and to the generators $K_{0,+,-}$ of the group SU(1, 1) the operators $a$, $a^+$, respectively $\boldsymbol{K}_{0,+,-}$, where $(a^+)^+ = a$, $\boldsymbol{K}_0^+ = \boldsymbol{K}_0$, $\boldsymbol{K}_\pm^+ = \boldsymbol{K}_\mp$. We impose to the cyclic vector $e_0$ to verify simultaneously the conditions

$$ae_0 = 0, \tag{6a}$$

$$\boldsymbol{K}_- e_0 = 0, \tag{6b}$$

$$\boldsymbol{K}_0 e_0 = k e_0; \quad k > 0, \quad 2k = 2, 3, \ldots \tag{6c}$$

We have considered in (6c) the positive discrete series representations $D_k^+$ of SU(1, 1) [35].

Perelomov's CS vectors associated to the group $G_1^J$ with Lie algebra the Jacobi algebra (3), based on the manifold $M$,

$$M \ := H_1/\mathbb{R} \times \mathrm{SU}(1,1)/\mathrm{U}(1), \tag{7a}$$

$$M = \mathcal{D}_1^J := \mathbb{C} \times \mathcal{D}_1, \tag{7b}$$

are defined as

$$e_{z,w} := \mathrm{e}^{za^+ + w\boldsymbol{K}_+} e_0, \quad z, w \in \mathbb{C}, \quad |w| < 1. \tag{8}$$

The displacement operator

$$D(\alpha) := \exp(\alpha a^+ - \bar{\alpha} a) = \exp\left(-\frac{1}{2}|\alpha|^2\right)\exp(\alpha a^+)\exp(-\bar{\alpha} a), \tag{9}$$

verifies the relation

$$D(\alpha_2) D(\alpha_1) = \mathrm{e}^{i\Im(\alpha_2 \cdot \bar{\alpha}_1)} D(\alpha_2 + \alpha_1). \tag{10}$$

Let us denote by $S$ – *the unitary squeezed operator* – the $D_+^k$ representation of the group SU(1, 1) and let us introduce the notation $\underline{S}(z) = S(w)$, where $w$ and $z$, $w \in \mathbb{C}$, $|w| < 1$, are related by (11c), (11d):

$$\underline{S}(z) := \exp(z\boldsymbol{K}_+ - \bar{z}\boldsymbol{K}_-), \quad z \in \mathbb{C}; \tag{11a}$$

$$S(w) = \exp(w\boldsymbol{K}_+)\exp(\eta \boldsymbol{K}_0)\exp(-\bar{w}\boldsymbol{K}_-); \tag{11b}$$

$$w = w(z) = \frac{z}{|z|}\tanh(|z|), \quad w \in \mathbb{C}, \quad |w| < 1; \tag{11c}$$

$$z = z(w) = \frac{w}{|w|}\operatorname{arctanh}(|w|) = \frac{w}{2|w|}\log\frac{1+|w|}{1-|w|}; \tag{11d}$$

$$\eta = \log(1 - w\bar{w}) = -2\log(\cosh(|z|)). \tag{11e}$$

We introduce also the normalized (squeezed) CS vector [34])

$$\Psi_{\alpha,w} := D(\alpha) S(w) e_0. \tag{12}$$

We introduce the auxiliary operators:

$$\boldsymbol{K}_+ = \frac{1}{2}(a^+)^2 + \boldsymbol{K}'_+, \tag{13a}$$

$$\boldsymbol{K}_- = \frac{1}{2}a^2 + \boldsymbol{K}'_-, \tag{13b}$$

$$\boldsymbol{K}_0 = \frac{1}{2}(a^+ a + \frac{1}{2}) + \boldsymbol{K}'_0, \tag{13c}$$

which have the properties

$$\boldsymbol{K}'_- e_0 = 0, \tag{14a}$$

$$\boldsymbol{K}'_0 e_0 = k' e_0; \quad k = k' + \frac{1}{4}; \tag{14b}$$

$$\left[\boldsymbol{K}'_\sigma, a\right] = \left[\boldsymbol{K}'_\sigma, a^+\right] = 0, \quad \sigma = \pm, 0, \tag{15a}$$

$$\left[\boldsymbol{K}'_0, \boldsymbol{K}'_\pm\right] = \pm \boldsymbol{K}'_\pm; \quad \left[\boldsymbol{K}'_-, \boldsymbol{K}'_+\right] = 2\boldsymbol{K}'_0. \tag{15b}$$

We recall the orthonormal system of coherent states associated to the group SU(1, 1):

$$e_{k,k+m} := a_{km}(\boldsymbol{K}_+)^m e_{k,k}; \quad a_{km}^2 = \frac{\Gamma(2k)}{m!\Gamma(m+2k)}, \tag{16}$$

and to the Heisenberg-Weyl group

$$\left|n\right\rangle = (n!)^{-\frac{1}{2}}(a^+)^n \left|0\right\rangle; \quad \left\langle n', n\right\rangle = \delta_{nn'}. \tag{17}$$

We write down the vector $e_0$ in (6) as

$$e_0 = e_0^H \otimes e_0^{K'}, \quad \text{where } \left|0\right\rangle \equiv e_0^H; \quad e_0^{K'} \equiv e_{k',k'}. \tag{18}$$

**Proposition 1.** *Perelomov's coherent state vector* (8) *can be written down as*

$$e_{z,w} = E(z,w) e_0^H \otimes \mathrm{e}^{w\boldsymbol{K}'} e_0^{K'} \tag{19}$$

*where the operator*

$$E(z,w) = \mathrm{e}^{za^+ + \frac{w}{2}(a^+)^2} \tag{20}$$

*has the action*

$$E(z,w)\mathrm{e}_0^H = \sum \frac{1}{(n!)^{1/2}}\left(-\frac{w}{2}\right)^{n/2} H_n\left(\frac{iz}{\sqrt{2w}}\right)\left|n\right\rangle, \tag{21}$$

*and $H_n$ are the Hermite polynomials*

$$H_n(x) = n! \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^m (2x)^{n-2m}}{m!(n-2m)!}. \tag{22}$$

*The second term in* (19) *admits the standard expansion in the operators* (16)

$$\mathrm{e}^{w\boldsymbol{K}'_+} e_0^{K'} = \sum \frac{w^m e_{k',k'+m}}{m! a_{k'm}}. \tag{23}$$

*The base of functions associated to the CS-vectors attached to the Jacobi group* $G_1^J$ *, based on the manifold M* (7b),

$$f_{|n\rangle;e_{k',k'+m}}(z,w):=\left(e_{\bar{z},\bar{w}},|n\rangle e_{k',k'+m}\right),\quad z\in\mathbb{C},\quad |w|<1, \tag{24}$$

*consists of the functions*

$$f_{|n\rangle;\,e_{k',k'+m}}(z,w)=f_{e_{k',k'+m}}(w)\frac{P_n(z,w)}{\sqrt{n!}}, \tag{25}$$

*where*

$$f_{e_{k,k+n}}(w):=(e_{z=0,\bar{w}},e_{k,k+n})=\sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}}w^n,\quad |w|<1. \tag{26}$$

*is the base of of holomorphic functions on* $\mathcal{D}_1$*, and the holomorphic polynomials* $P_n(z,w)$ *have the expression*

$$P_n(z,w)=n!\sum_{k=0}^{[\frac{n}{2}]}\left(\frac{w}{2}\right)^k\frac{z^{n-2k}}{k!(n-2k)!}. \tag{27}$$

*Let* $K=K(\bar{z},\bar{w},z,w)$*, where* $z\in\mathbb{C}$*,* $w\in\mathbb{C}$*,* $|w|<1$*,*

$$K:=(e_0,e^{\bar{z}a+\bar{w}\boldsymbol{K}_-}e^{za^++w\boldsymbol{K}_+}e_0). \tag{28}$$

*Then the reproducing kernel is*

$$K=(1-w\bar{w})^{-2k}\exp\frac{2z\bar{z}+z^2\bar{w}+\bar{z}^2w}{2(1-w\bar{w})}. \tag{29}$$

*More generally, the kernel* $K(z,w;\bar{z}',\bar{w}'):=(e_{\bar{z},\bar{w}},e_{\bar{z}',\bar{w}'}):\mathcal{D}_1^J\times\bar{\mathcal{D}}_1^J\to\mathbb{C}$ *is:*

$$K(z,w;\bar{z}',\bar{w}')=(1-w\bar{w}')^{-2k}\exp\frac{2\bar{z}'z+z^2\bar{w}'+\bar{z}'^2w}{2(1-w\bar{w}')}. \tag{30}$$

*The reproducing kernel* (30) $K:M\times\bar{M}\to\mathbb{C}$ *admits the series expansion in the base functions* (25)

$$K(z,w;\bar{z},\bar{w}')=\sum_{n,m}f_{|n\rangle,e_{k',k'+m}}(z,w)\bar{f}_{|n\rangle,e_{k',k'+m}}(z',w'). \tag{31}$$

*The normalized squeezed state vector* (12) *and the un-normalized (Perelomov's coherent state) vector* (8) *are related by the relation*

$$\Psi_{\alpha,w}=(1-w\bar{w})^k\exp\left(-\frac{\bar{\alpha}}{2}z\right)e_{z,w}, \tag{32}$$

*where* $z=\alpha-w\bar{\alpha}$.

## 4. THE RESOLUTION OF UNITY

From first part of the next proposition we can see the holomorphic action of the Jacobi group

$$G_1^J := H_1 \rtimes \mathrm{SU}(1,1) \qquad (33)$$

on the manifold $\mathcal{D}_1^J$ (7b). For self-contentedness, we also reproduce the resolution of unity:

**Proposition 2.** *Let us consider the action* $S(g)D(\alpha)e_{z,w}$, *where* $g \in \mathrm{SU}(1,1)$ *has the form* (34),

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad \textit{where} \quad |a|^2 - |b|^2 = 1. \qquad (34)$$

$D(\alpha)$ *is given by* (9), *and Perelomov's coherent state vector is defined in* (8). *Then we have the formula* (35) *and the relations* (36), (37)–(39) *below:*

$$S(g)D(\alpha)e_{z,w} = \lambda e_{z_1,w_1}, \quad \lambda = \lambda(g, \alpha; z, w), \qquad (35)$$

$$z_1 = \frac{\alpha - \bar{\alpha} w + z}{\bar{b} w + \bar{a}}; \quad w_1 = g \cdot w = \frac{aw + b}{\bar{b} w + \bar{a}}, \qquad (36)$$

$$\lambda = (\bar{a} + \bar{b} w)^{-2k} \exp\left(\frac{z}{2}\bar{\alpha}_0 - \frac{z_1}{2}\bar{\alpha}_2\right) \exp i\,\Im(\alpha \cdot \bar{\alpha}_0), \qquad (37)$$

$$\alpha_0 = \frac{z + \bar{z} w}{1 - w\bar{w}}, \qquad (38)$$

$$\alpha_2 = (\alpha + \alpha_0)a + (\bar{\alpha} + \bar{\alpha}_0)b. \qquad (39)$$

*The scalar product of functions from the space* $\mathcal{F}_K$ *corresponding to the kernel defined by* (30) *on the manifold* (7b) *is:*

$$(\phi, \psi) = \Lambda \int_{z \in \mathbb{C}; |w|<1} \bar{f}_{\phi}(z, w) f_{\psi}(z, w)(1 - w\bar{w})^{2k} \exp - \frac{|z|^2}{1 - w\bar{w}} \exp - \frac{z^2 \bar{w} + \bar{z}^2 w}{2(1 - w\bar{w})} d\nu,$$

*where the value of the* $G^J$*-invariant measure* $d\nu$ *is*

$$d\nu = \frac{d\Re w\, d\Im w}{(1 - w\bar{w})^3} d\Re z\, d\Im z \qquad (40)$$

*and*

$$\Lambda = \frac{4k - 3}{2\pi^2}. \qquad (41)$$

*The Kähler potential is calculated as the logarithm of the reproducing kernel* (30), $f := \log K$, *i.e.*

$$f = \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2 w}{2(1 - w\bar{w})} - 2k\log(1 - w\bar{w}), \tag{42}$$

*and the Kähler two-form* $\omega_1$ *is obtained via the formula:*

$$-i\,\omega_1 = f_{z\bar{z}}dz \wedge d\bar{z} + f_{z\bar{w}}dz \wedge d\bar{w} - f_{\bar{z}w}d\bar{z} \wedge dw + f_{w\bar{w}}dw \wedge d\bar{w}, \tag{43}$$

$$-i\,\omega_1 = \frac{2k}{(1 - w\bar{w})^2}dw \wedge d\bar{w} + \frac{A \wedge \bar{A}}{1 - w\bar{w}}, \quad A = dz + \bar{\alpha}_0 dw, \quad \alpha_0 = \frac{z + \bar{z}w}{1 - w\bar{w}}. \tag{44}$$

## 5. KÄHLER-BERNDT'S APPROACH

Rolf Berndt – alone or in collaboration – has studied the real Jacobi group $G^J(\mathbb{R})$ in several references, from which I mention [23, 36, 24, 37]. The Jacobi group appears (see explanation in [38]) in the context of the so called *Poincaré group* or *The New Poincaré group* – the double cover of the de Sitter group $SO_0(4, 1)$ – investigated by Erich Kähler as the 10-dimensional group $G^K$ which invariates a hyperbolic metric [25–27]. Kähler and Berndt have investigated the Jacobi group $G_0^J(\mathbb{R}) := SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ acting on the manifold $\mathfrak{X}_1^J := \mathcal{H}_1 \times \mathbb{C}$, where $\mathcal{H}_1$ is the upper half plane $\mathcal{H}_1 := \{v \in \mathbb{C} \mid \Im(v) > 0\}$.

In Remarks 1 and 2 below, we proof two results from [24], which we need in order two express the two-form $\omega_1$ in the coordinates used by Kähler and Berndt. The main ingredient in the proof of Remark 1 below is the Iwasawa decomposition. Let us mention that Iwasawa decomposition was largely used in applications in Optics, see *e.g.* [39, 15]. Note also that the linear fractional transformation expressed by the first relation (45) is the famous “abcd law” in Optics [40, 28].

**Remark 1.** *The action of* $G_0^J(\mathbb{R})$ *on* $\mathfrak{X}_1^J$ *is given by* $(g, (v, z)) \to (v_1, z_1)$, $g = (M, l)$, *where*

$$v_1 = \frac{av + b}{cv + d}, \quad z_1 = \frac{z + l_1 v + l_2}{cv + d}; \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}),\ (l_1, l_2) \in \mathbb{R}^2. \tag{45}$$

*Proof.* Let us use the notation of [24]. We denote $G^J(\mathbb{R}) := SL_2(\mathbb{R}) \rtimes H(\mathbb{R})$, where here $H(\mathbb{R})$ denotes the real HW group with the composition law:

$$(\lambda, \mu, \kappa)(\lambda', \mu', \kappa') = \left(\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \begin{vmatrix} X \\ X' \end{vmatrix}\right), \quad \begin{vmatrix} X \\ X' \end{vmatrix} = \det\begin{pmatrix} X \\ X' \end{pmatrix}. \tag{46}$$

If $g=(M,X,\kappa)\in G^J(\mathbb{R})$, where $M\in \mathrm{SL}_2(\mathbb{R})$, $X=(\lambda,\mu)$, $(X,\kappa)\in\mathbb{R}^3$, then the composition law in the real Jacobi group is

$$gg'=\left(MM', XM'+X', \kappa+\kappa'+\begin{vmatrix}XM'\\X'\end{vmatrix}\right). \quad (47)$$

The action of $G^J(\mathbb{R})$ over the $H(\mathbb{R})$ is

$$M(X,\kappa)M^{-1}=(XM^{-1},\kappa). \quad (48)$$

Let us consider the Iwasawa decomposition for a matrix $M\in\mathrm{SL}_2(\mathbb{R})$:

$$M=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}, \quad y>0. \quad (49)$$

If

$$M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}, \quad (50)$$

then we find for $x$, $y$, $\theta$ in (49)

$$x=\frac{ac+bd}{d^2+c^2};\quad y=\frac{1}{d^2+c^2};\quad \sin\theta=-\frac{c}{\sqrt{c^2+d^2}};\quad \cos\theta=\frac{d}{\sqrt{c^2+d^2}}. \quad (51)$$

For $g=(M,X,\kappa)\in G^J(\mathbb{R})$, the EZ-coordinates (Eichler-Zagier, cf. the definition at p. 12 and p. 51 in [24]) are $(x, y, \theta, \lambda, \mu, \kappa)$. Let $\tau=x+iy\in\mathcal{H}_1$, $z=\xi+i\eta=p\tau+q$, where

$$(p,q)=XM^{-1}=(\lambda d-\mu c, -\lambda b+\mu a). \quad (52)$$

If we attache a “$_*$” to the results of elements of the composition rule (47), we have

$$x_*=\frac{AC+BD}{D^2+C^2};\quad y_*=\frac{1}{D^2+C^2}, \quad (53)$$

where

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}. \quad (54)$$

We find out:

$$D^2+C^2=c^2(a'^2+b'^2)+d^2(c'^2+d'^2)+2cd(a'c'+b'd'),$$

*i.e.*

$$D^2+C^2=c^2(a'^2+b'^2)+\frac{d^2}{y'}+2cd\frac{x'}{y'}.$$

Similarly,

$$AC+BD=ac(a'^2+b'^2)+(ad+bc)\frac{x'}{y'}+\frac{bd}{y'}.$$

We find for $\tau_*=x_*+iy_*$

$$\tau_*=\frac{ac(a'^2+b'^2)y'+(ad+bc)x'+iy'+bd}{c^2(a'^2+b'^2)y'+2cdx'+d^2}. \tag{55}$$

Let us verify the first relation (45), in the present notation

$$\tau_*=\frac{a\tau'+b}{c\tau'+d}, \tag{56}$$

where

$$\tau'=x'+\mathrm{i}y'=\frac{a'c'+b'd'+\mathrm{i}}{d'^2+c'^2}. \tag{57}$$

Combining (56), (57), we find out

$$\tau_*=\frac{(ax'+b)(cx'+d)+acy'^2+iy'}{(cx'+d)^2+c^2y'^2}, \tag{58}$$

and we have to verify the identify (55) and (58).

In order to prove the second equation (45), we calculate firstly

$$(P_*,Q_*)=(LD-MC,-LB+MA),$$

where

$$(L,M)=(\lambda'+\lambda a'+\mu c',\mu'+\lambda b'+\mu d'),$$

and we find

$$P_*=\lambda'(cb'+dd')-\mu'(ca'+dc')+\lambda d-\mu c; \tag{59a}$$

$$Q_*=-\lambda'(ab'+bd')+\mu'(ad'+bc')-\lambda b+\mu a. \tag{59b}$$

Then we obtain

$$z_*:=P_*\tau_*+Q_*=\frac{(P_*a+Q_*c)\tau'+P_*b+Q_*a}{c\tau'+d}.$$

The nominator $E$ of the last expression of $\tau^*$ should be identified with $E=p'\tau'+q'+\lambda\tau'+\mu$, *i.e.* it remains to verify that

$$P_*a+Q_*b=p'+\lambda;$$

$$P_*b+Q_*d=q'+\mu'.$$

In conclusion, using the multiplication law (47), the Iwasawa decomposition (49) and the equations (51), (52), we have obtained the action of $G^J(\mathbb{R})$ on the base $\mathfrak{X}_1^J$

$$g(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right), \tag{60}$$

and Remark 1 is proved.

Let us now recall that

$$C^{-1}\mathrm{SL}_2(\mathbb{R})C = \mathrm{SU}(1,1), \quad \text{where } C = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}. \tag{61}$$

If $M \in \mathrm{SL}_2(\mathbb{R})$ is the matrix (50), then, under the transformation (61)

$$M_* = C^{-1}MC = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 - |\beta|^2 = 1, \tag{62}$$

where

$$2\alpha = a + d + i(b - c); \quad 2\beta = a - d - i(b + c). \tag{63}$$

Now we pass to the complex group $G^J_{\mathbb{C}} = C^{-1}G^J(\mathbb{R})C$. Under the transformation (61), $g = (M, X, \kappa) \in \mathrm{SL}_2(\mathbb{R}) \ltimes H(\mathbb{R})$ is twisted to $g_* = (M_*, X_*, \kappa)$, where $M_*$ is given by (62), while, due to action (48), $X_* = XC = (i\lambda - \mu, i\lambda + \mu)$.

Also the map (61) induces a transformation of the bounded domain $\mathcal{D}_1$ into the upper half plane $\mathcal{H}_1$ and

$$\tau \in \mathcal{H}_1 \mapsto \tau_* = C^{-1}(\tau) = \frac{\tau - i}{\tau + i} \in \mathcal{D}_1. \tag{64}$$

The action $C^{-1}G^J_0(\mathbb{R})C$ descends on the basis to the biholomorphic map: $\check{C}^{-1} : \mathfrak{X}^J_1 := \mathcal{H}_1 \times \mathbb{C} \to \mathcal{D}^J_1 := \mathcal{D}_1 \times \mathbb{C}: \ (\tau, z) \mapsto (\tau_*, z_*)$. Here $\tau_*$ is given by (64), while $z_* = p_*\tau_* + q_*$. So, $(p, q) = (\lambda, \mu)M^{-1}$, and $(p_*, q_*) = (\lambda_*, \mu_*)M_*^{-1}$. But $M_* = C^{-1}MC$, and $(p_*, q_*) = (p, q)C = (-q + ip, q + ip)$, and we get $z_* = \dfrac{2iz}{\tau + i}$.

In a different notation, we have shown that

**Remark 2.** *The action* $C^{-1}G^J_0(\mathbb{R})C$, *descends on the basis to the biholomorphic map:* $\check{C}^{-1} : \mathfrak{X}^J_1 := \mathcal{H}_1 \times \mathbb{C} \to \mathcal{D}^J_1 := \mathcal{D}_1 \times \mathbb{C}:$

$$w = \frac{v - i}{v + i}; \quad z = \frac{2iu}{v + i}, \quad w \in \mathcal{D}_1, \quad v \in \mathcal{H}_1, \quad z \in \mathbb{C}. \tag{65}$$

The $G^J_0(\mathbb{R})$ -invariant closed two-form considered by Kähler-Berndt is:

$$\omega'_1 = \alpha\frac{dv \wedge d\bar{v}}{(v - \bar{v})^2} + \beta\frac{1}{v - \bar{v}}B \wedge \bar{B}, \quad B = du - \frac{u - \bar{u}}{v - \bar{v}}dv, \quad v, u \in \mathbb{C}, \quad \Im(v) > 0, \tag{66}$$

cf. §36 in [27]; see also §3.2 in [23].

Under the mapping (65), the two-form $\omega_1$ (44) reads

$$-i\,\omega_1' = -\frac{2k}{(\bar{v}-v)^2}dv\wedge d\bar{v} + \frac{2}{i(\bar{v}-v)}B\wedge\bar{B}, \tag{67}$$

*i.e.* (66). In fact, we have proved that

**Remark 3.** *When expressed in the coordinates* $(v,u)\in\mathfrak{X}_1^J$ *which are related to the coordinates* $(w,z)\in\mathcal{D}_1^J$ *by the map* (65) *given by* Remark 2, *the Kähler two-form* (44) *is identical with the one* (67) *considered by Kähler-Berndt.*

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