

PHYSICAL COORDINATES AS DYNAMIC VARIABLES FOR THE SUPERPARTICLE FROM ITS GEOMETRICAL ACTION

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Abstract. We show that the particle actions in the superspace that are invariant with respect to general covariance transformations can be formulated in terms of physical coordinates with non zero evolution Hamiltonians by identifying these coordinates with some dynamic variables. The local κ -symmetry for superparticle actions in this formulation is briefly discussed.

Key words: superparticle, Hamiltonian formulation, relativistic theories.

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1. INTRODUCTION AND SUMMARY

It is known [1–5] that the reparameterization invariance of the theories of relativistic particles and relativistic strings, as well as the invariance of the gravity theory with respect to general covariance transformations, results in serious problems when analyzing these theories in the Hamiltonian formalism : occurs the nullification of the Hamiltonian because of this invariance.

For the relativistic particle, this problem was already circumvented in the first Einstein and Poincare' papers [6, 7] thanks to identifying one of the dynamical variables, $x_0(\tau)$, with the physical time, and this identification is natural in the special relativity theory.

The idea that the time measured by an observer's clock is a dynamic variable seems rather strange in field theories, where the fields are functions of space and time. In the string theory, time was considered a dynamic variable in [8, 9], which allows relating this theory with the Born-Infeld theory.

It is possible to give a correct description of these reparameterizations invariant systems, from a dynamical point of view, passing to the physical variables

by means of the Levi-Civita canonical transformations, as was shown in [10, 11]. These canonical transformations make the dynamical system under consideration suitable to be integrated or quantized.

Strongly motivated to extend the above concepts to a toy superspace we apply the Levi-Civita canonical transformations to the simple model of superparticle of Volkov and Pashnev [12, 13], that is the type G4 in the description of Casalbuoni [14, 15] to obtain the unconstrained form of the superparticle action, after that these canonical transformations have been performed. This final unconstrained action, that is in function on the physical variables, is suitable to be quantized or integrated. Recently in [18, 19], the importance of the constraints in the superparticle actions when local supersymmetry transformations are introduced, was shown. The space-time covariant formulation of super p-branes is known to have a local fermionic invariance on the world manifold, first discovered by Siegel [21] for the superparticle and posteriorly in [22] for superstrings. This invariance helps to balance the number of commuting and anti-commuting degrees of freedom mainly in models where with the boson and fermion variables belonging to different representations of the Lorentz group of the target space-time. As the parameter of this transformation is an anti-commuting space-time spinor κ varying in an arbitrary way over the world manifold. In this sense this κ -invariance is a supersymmetry. The another motivation of this paper is to discuss shortly what happens with the global SUSY and this local κ -supersymmetry when these Levi-Civita canonical transformations are performed in the superparticle-model under consideration because it is well established that the actions that are κ -invariants have the physical interpretation as the leading term in the effective action describing the the low energy dynamics of topological defects of supersymmetric field theories [23–28].

2. THE SUPERPARTICLE MODEL

In the superspace the coordinates are given not only by the spacetime x_μ coordinates, but also for anticommuting spinors θ^α and $\bar{\theta}^{\dot{\alpha}}$. The resulting metric [12, 13] must be invariant to the action of the Poincare group, and invariant also to the supersymmetry transformations

$$x'_\mu = x_\mu + i\left(\theta^\alpha(\sigma)_{\alpha\beta}\bar{\xi}^{\dot{\beta}} - \xi^\alpha(\sigma)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right); \quad \theta'^\alpha = \theta^\alpha + \xi^\alpha; \quad \bar{\theta}'^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}} + \bar{\xi}^{\dot{\alpha}}$$

The simplest super-interval that obeys the requirements of invariance given above, is the following

$$ds^2 = \omega_\mu \omega^\mu + a \omega_\alpha \omega^\alpha - a^* \omega_{\dot{\alpha}} \omega^{\dot{\alpha}} \quad (1)$$

where

$$\omega_\mu = dx_\mu - i(d\theta\sigma_\mu\bar{\theta} - \theta\sigma_\mu d\bar{\theta}); \quad \omega^\alpha = d\theta^\alpha; \quad \omega^{\dot{\alpha}} = d\bar{\theta}^{\dot{\alpha}}$$

are the Cartan forms of the group of supersymmetry [17].

The spinorial indexes are related as follows

$$\theta^\alpha = \varepsilon^{\alpha\beta}\theta_\beta; \quad \theta_\alpha = \theta^\beta\varepsilon_{\beta\alpha}; \quad \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}; \quad \varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}; \quad \varepsilon_{12} = \varepsilon^{12} = 1$$

and of analog manner for the spinors with punctuated indexes. The complex constants a and a^* in the line element (1) are arbitrary. This arbitrariness for the choice of a and a^* are constrained by the invariance and reality of the interval (1).

As we have extended our manifold to include fermionic coordinates, it is natural to extend also the concept of trajectory of point particle to the superspace. To do this we take the coordinates $x(\tau)$, $\theta(\tau)$ and $\bar{\theta}^{\dot{\alpha}}(\tau)$ depending on the evolution parameter τ . Geometrically, the function action that will describe the world-line of the superparticle, is

$$S = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{\overset{\circ}{\omega}_\mu \overset{\circ}{\omega}^\mu + a \dot{\theta}_\alpha \dot{\theta}^\alpha - a^* \dot{\bar{\theta}}_{\dot{\alpha}} \dot{\bar{\theta}}^{\dot{\alpha}}} = \int_{\tau_1}^{\tau_2} d\tau L(x, \theta, \bar{\theta}) \quad (2)$$

where $\overset{\circ}{\omega}_\mu = \dot{x}_\mu - i(\dot{\theta}\sigma_\mu\bar{\theta} - \theta\sigma_\mu\dot{\bar{\theta}})$ and the upper point means derivative with respect to the parameter τ , as is usual.

The momenta, canonically conjugated to the coordinates of the superparticle, are

$$\begin{aligned} \mathcal{P}_\mu &= \partial L / \partial x^\mu = (m^2/L) \overset{\circ}{\omega}_\mu \\ \mathcal{P}_\alpha &= \partial L / \partial \dot{\theta}^\alpha = i\mathcal{P}_\mu (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} + (m^2 a/L) \dot{\theta}_\alpha \\ \mathcal{P}_{\dot{\alpha}} &= \partial L / \partial \dot{\bar{\theta}}^{\dot{\alpha}} = i\mathcal{P}_\mu \theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} - (m^2 a/L) \dot{\bar{\theta}}_{\dot{\alpha}} \end{aligned} \quad (3)$$

It is difficult to study this system in the Hamiltonian formalism framework because of the constraints and the nullification of the Hamiltonian. As the action (2) is invariant under reparametrizations of the evolution parameter

$$\tau \rightarrow \tilde{\tau} = f(\tau) \quad (4)$$

one way to overcome this difficulty is to make the dynamic variable x_0 the time. For this, it is sufficient to use the chain rule of derivatives (with special care of the anticommuting variables)¹ and to write the action in the form

¹ We take the Berezin convention for the Grassmannian derivatives: $\delta F(\theta) = \frac{\partial F}{\partial \theta} \delta \theta$.

$$S = -m \int_{\tau_1}^{\tau_2} \dot{x}_0 d\tau \sqrt{[1 - iW_{,0}^0]^2 - [x^i - W_{,0}^i]^2} + a \dot{\theta}_\alpha \dot{\theta}^\alpha - a^* \dot{\bar{\theta}}_{\dot{\alpha}} \dot{\bar{\theta}}^{\dot{\alpha}} \quad (5)$$

where the $W_{,0}^\mu$ was defined by

$$\begin{aligned} \overset{\circ}{\omega}{}^0 &= \dot{x}^0 [1 - iW_{,0}^0] \\ \overset{\circ}{\omega}{}^i &= \dot{x}^0 [x_{,0}^i - iW_{,0}^i] \end{aligned} \quad (6)$$

whence $x_0(\tau)$ turns out to be the evolution parameter

$$S = -m \int_{x_0(\tau_1)}^{x_0(\tau_2)} dx_0 \sqrt{[1 - iW_{,0}^0]^2 - [x^i - W_{,0}^i]^2} + a \dot{\theta}_\alpha \dot{\theta}^\alpha - a^* \dot{\bar{\theta}}_{\dot{\alpha}} \dot{\bar{\theta}}^{\dot{\alpha}} \equiv \int dx_0 L \quad (7)$$

Physically this parameter (we call it the dynamical parameter) is the time measured by an observer's clock in the rest frame.

Therefore, the invariance of a theory with respect to the invariance of the coordinate evolution parameter means that one of the dynamic variables of the theory ($x_0(\tau)$ in this case) becomes the observed time with the corresponding non-zero Hamiltonian

$$\begin{aligned} H &= \mathcal{P}_\mu \dot{x}^\mu + \Pi_\alpha \dot{\theta}^\alpha + \Pi_{\dot{\alpha}} \dot{\bar{\theta}}^{\dot{\alpha}} - L = \\ &= \sqrt{m^2 - \left(\mathcal{P}_\mu \mathcal{P}^\mu + \frac{1}{a} \Pi_\alpha \Pi^\alpha - \frac{1}{a^*} \Pi_{\dot{\alpha}} \Pi^{\dot{\alpha}} \right)} \end{aligned} \quad (8)$$

where

$$\begin{aligned} \Pi_\alpha &= \mathcal{P}_\alpha + i \mathcal{P}_\mu (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \\ \Pi_{\dot{\alpha}} &= \mathcal{P}_{\dot{\alpha}} - i \mathcal{P}_\mu \theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \end{aligned}$$

Choosing $x_0(\tau)$ as the evolution parameter, we thus fix the reference frame. This procedure to fix the reference frame is called *the physical realization of the relativistic particle* [11]. When a specific physical realization is chosen, we lose all other realizations. In particular, the co-moving frame in which the time is the proper time with the interval

$$dt = \sqrt{\frac{ds^2}{d\tau^2}} d\tau \quad (9)$$

But, it is easy to show that the relativistic action (2), written in the invariant form with the additional variable $e(\tau)$ (einbein)

$$\begin{aligned}
S &= -\frac{m}{2} \int_{\tau_1}^{\tau_2} d\tau \left[\frac{1}{e} \left(\overset{\circ}{\omega}_\mu \overset{\circ}{\omega}^\mu + a \dot{\theta}_\alpha \dot{\theta}^\alpha - a^* \dot{\bar{\theta}}_{\dot{\alpha}} \dot{\bar{\theta}}^{\dot{\alpha}} \right) + e \right] \\
&= \int d\tau \left[\mathcal{P}_\mu \mathcal{P}^\mu + \frac{1}{a} \Pi_\alpha \Pi^\alpha - \frac{1}{a^*} \Pi_{\dot{\alpha}} \Pi^{\dot{\alpha}} \right] - \\
&\quad - \frac{e}{2m} \left[m^2 - \left(\mathcal{P}_\mu \mathcal{P}^\mu + \frac{1}{a} \Pi_\alpha \Pi^\alpha - \frac{1}{a^*} \Pi_{\dot{\alpha}} \Pi^{\dot{\alpha}} \right) \right]
\end{aligned} \tag{10}$$

also describes the relativistic particle in the co-moving frame. The equations of motion for the action (11) are

$$\begin{aligned}
m^2 - \left(\mathcal{P}_\mu \mathcal{P}^\mu + \frac{1}{a} \Pi_\alpha \Pi^\alpha - \frac{1}{a^*} \Pi_{\dot{\alpha}} \Pi^{\dot{\alpha}} \right) &= 0 \\
\frac{\partial H}{\partial \Pi} &= \dot{\theta} \quad (\text{or } \dot{\bar{\theta}}) \quad \frac{\partial H}{\partial \mathcal{P}} = \dot{x}
\end{aligned} \tag{11}$$

Equations (12) contain two times: x_0 is the time in the rest frame and t is the time in the co-moving frame.

The relation

$$x_0 = \frac{\mathcal{P}_0}{m} t \tag{12}$$

between these two times describes the purely relativistic effect of changing the time when passing to another reference frame.

As shown in [12], there exists a scenario of a dynamic transition to the co-moving frame using the Levi-Civita canonical transformation $(\mathcal{P}_\mu, \Pi_\alpha, \Pi_{\dot{\alpha}}; x_\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \rightarrow (P_\mu, P_\alpha, P_{\dot{\alpha}}; Q_\mu, Q_\alpha, Q_{\dot{\alpha}})$ [10, 11]

$$\begin{aligned}
P_0 &= \frac{1}{2m} \left(\mathcal{P}_\mu \mathcal{P}^\mu + \frac{1}{a} \Pi_\alpha \Pi^\alpha - \frac{1}{a^*} \Pi_{\dot{\alpha}} \Pi^{\dot{\alpha}} \right), \quad P_i = \mathcal{P}_i, \quad P_\alpha = \Pi_\alpha, \quad P_{\dot{\alpha}} = \Pi_{\dot{\alpha}}, \\
Q_0 &= x_0 \frac{m}{\mathcal{P}_0}, \quad Q_i = x_i + \frac{x_0}{\mathcal{P}_0} \mathcal{P}_i, \quad Q_\alpha = \theta_\alpha + \frac{x_0}{\mathcal{P}_0 a} \Pi_\alpha, \quad Q_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}} - \frac{x_0}{\mathcal{P}_0 a^*} \Pi_{\dot{\alpha}}
\end{aligned} \tag{13}$$

which transforms the constraint into the new momentum P_0 and the time x_0 into the proper time (10). Indeed we can use the eqs. (14) to express the old momenta \mathcal{P}_μ , Π_α , $\Pi_{\dot{\alpha}}$ and coordinates x_μ , θ_α , $\bar{\theta}_{\dot{\alpha}}$ through the new ones as

$$\begin{aligned}
\mathcal{P}_0 &= \pm \sqrt{2mP_0 - \left(P_i P^i + \frac{1}{a} P_\alpha P^\alpha - \frac{1}{a^*} P_{\dot{\alpha}} P^{\dot{\alpha}} \right)}; \quad \mathcal{P}_i = P_i \\
x_0 &= \pm \frac{Q_0}{m} \sqrt{2mP_0 - \left(P_i P^i + \frac{1}{a} P_\alpha P^\alpha - \frac{1}{a^*} P_{\dot{\alpha}} P^{\dot{\alpha}} \right)}
\end{aligned}$$

$$x_i = Q_i + \frac{Q_0}{m} P_i \quad \theta_\alpha = Q_\alpha + \frac{Q_0}{ma} P_\alpha \quad \bar{\theta}_{\dot{\alpha}} = Q_{\dot{\alpha}} - \frac{Q_0}{ma^*} P_{\dot{\alpha}} \quad (14)$$

the action (7) in the new variables then becomes

$$S = \int_{\tau_1}^{\tau_2} \left[P_\mu \dot{Q}^\mu + P_\alpha \dot{Q}^\alpha + P_{\dot{\alpha}} \dot{Q}^{\dot{\alpha}} - e \left(P_0 - \frac{m}{2} \right) + \frac{d(tP_0)}{dt} \right] \quad (15)$$

Varying the action (16) with respect to P_0 , we define the new variable $Q_0 = t$

$$\frac{dQ_0}{d\tau} = e(\tau) \quad (16)$$

to be the new proper time (10), and varying (16) with respect to $e(\tau)$ we obtain the constraint

$$P_0 - \frac{m}{2} = 0 \quad (17)$$

Finally, resolving this constraint with respect to the momentum component P_0 , we obtain the expression

$$S = \int_{t_1}^{t_2} \left[P_i \frac{dQ^i}{dt} + P_\alpha \frac{dQ^\alpha}{dt} + P_{\dot{\alpha}} \frac{dQ^{\dot{\alpha}}}{dt} - \frac{m}{2} - \frac{d(tP_0)}{dt} \right] \quad (18)$$

Inverse Levi-Civita canonical transformations (14) and solutions (16) and (17) establish the relation between two reference frames, in this case in the super-space, with different physical realizations of the same particle. The reparametrization invariance therefore allows describing two physical realizations of the same particle by two constraint-free mechanics, while these mechanics are related through purely relativistic effects. In the dynamic transition given by the formulas (13), the global SUSY is preserved but the κ -invariance is not explicitly manifest in the expression (18). In order to restore the local relativistic κ -symmetry there are two possibilities to induce it in (18). These two ways are²

$$\mathcal{L}_A \equiv \mathcal{L} + \zeta D + \overline{\zeta} \overline{D} + eH$$

$$\mathcal{L}_B \equiv \mathcal{L} + \dot{\zeta} D + \overline{\dot{\zeta}} \overline{D} + eH$$

where only the second choice (*i.e.* \mathcal{L}_B) is the correct one³. It is interesting to note that these different ways to introduce the local supersymmetry was also obtained

² Here is the \mathcal{L} is the Lagrangian density, $\zeta, \dot{\zeta}(\overline{\zeta}, \overline{\dot{\zeta}})$ are Lagrange multipliers and $D(\overline{D})$ are the covariant derivatives, as usual.

³ It fact is easily seen from the point of view of the first order formulation of the superparticle action: when the metric (1) degenerates, this new Lagrangian is closest to the Siegel model.

and analyzed in ref. [20] taking as the starting point the functional approach of the the classical mechanics and the BRST formulation where was explicitly shown that are two gauge theories which differ in the gauge couplings: in the Lagrangian and in the physical Hilbert spaces.

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