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ON THE INDEPENDENT POINTS IN THE SKY FOR THE SEARCH OF PERIODIC GRAVITATIONAL WAVE

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Abstract. We investigate the independent points in the sky require to search the periodic gravitational wave, assuming the noise power spectral density to be flat. We have made an analysis with different initial azimuth of the Earth for a week data set. The analysis shows significant difference in the independent points in the sky for the search. We numerically obtain an approximate relation to make trade-off between computational cost and sensitivities. We also discuss the feasibility of the coherent search in small frequency band in reference to advanced LIGO.

Key words: gravitational wave, data analysis, periodic sources.

INTRODUCTION

The first generation of laser interferometric gravitational wave observatory (LIGO) [Abramovici (1992)] is in operation. However, its has been planned that by the end of the initial LIGO science run it would undergo upgrades to significantly improve the sensitivity. Now referred to as advanced LIGO (formerly LIGO II) [Weinstein, 2002]. The detector will sweep their broad quadrupolar beam pattern across the sky as the earth moves. Hence the data analysis system will have to carry out all sky searches for its sources. In this, search of the PGW without a priori knowledge appears to computationally quite demanding even by the standard computers expected to available in the near future. It appears that due to limited computational resource it will be not feasible to perform all sky all frequency search in the months/year data set. However, if advanced LIGO achieve its design sensitivity $\sim 10^{-23}$ or better (Weinstein, 2002), then it may be feasible to perform all sky search for a day to week data set in small frequency band for the sources emitting signal of amplitude $\gtrsim 10^{-26}$. The search of the potential sources may be more significant, if done in the frequency band where most of the Pulsars are detected by other means. Also, the choice of sophisticated, optimal data analysis methods and a clever programming is also integral part to search the signal buried in the noise with the available computation power.

The current status of the search indicates that its important to detect the gravitational waves (GW) rather finding the source location more accurately. Hence, one would like to do minimum Doppler correction or/and to make the search templates. In reference to the all sky search, Schutz (1991), has introduced the concept of patch in the sky as the region of space throughout which the required Doppler correction remains the same. He roughly estimated the number of patches required and shown that for 10^7 s observation data set of one KHz signal would be about 1.3×10^{13}. This also indicates the bank of templates require for the coherent all sky search. The size of the patch may be increased, hence reducing the number of points require for the Doppler correction, by manipulating the output of the detector. Which in turn demand the detail investigation of the parameters affecting the phase of the modulated signal. In this, the initial azimuth of the Earth play a vital role in the modulation of the signal, particularly for the analysis of day to week data set. Hence, in the next section, we incorporate the initial azimuth of the Earth in the Fourier transform (FT) obtained by Srivastava and Sahay (2002a,b) for arbitrary observation time.

The limited computation power make to search the signal in the short observation data set, may be a day/week, so in section 3, employing the concept of fitting factor (Apostolatos, 1995), we investigate the independent points in the sky ($N_{\rm sky}$) require for an all sky search for a week data set with different initial azimuth of the Earth, assuming the noise power spectral density to be flat. Also, we numerically obtain an approximate relation to make trade-off between computational cost and sensitivities. In section 4, we discuss the feasibility of the coherent search in reference to advanced LIGO. Section 5 contains the conclusions of the paper.

2. FOURIER TRANSFORM

The FT analysis of the frequency modulated PGW signal has been done by Srivastava and Sahay (2002a,b) by taking account the effects arising due to the rotational as well as orbital motion of the Earth. However, they have neglected an important parameter, the initial azimuth of the Earth, which affects significantly in the spacing of the parameters space for an all sky search. To obtain the FT with taking account the Earth initial azimuth, we rewrite the phase of the received PGW signal of frequency f_o at time t given by them and may be written as

$$\Phi(t) = 2\pi f_0 t + \mathcal{Z}\cos\left(a\xi_{rot} - \sigma\right) + \mathcal{N}\cos\left(a\xi_{rot} - \delta\right) - \mathcal{M}, \quad (1)$$

where:

$$\left.\begin{aligned}
&\mathcal{M} = \frac{2\pi f_o}{c}\left(R_{se}\sin\theta\cos\sigma\sqrt{P^2 + Q^2\cos\delta}\right),\\
&\mathcal{Z} = \frac{2\pi f_o}{c}R_{se}\sin\theta,\\
&\mathcal{N} = \frac{2\pi f_o}{c}\sqrt{P^2 + Q^2},\\
&\mathcal{P} = R_e\sin\alpha\left(\sin\theta\sin\phi\cos\varepsilon + \cos\theta\sin\varepsilon\right),\\
&\mathcal{Q} = R_e\sin\alpha\sin\theta\cos\phi,\\
&\sigma = \phi - \beta_{orb}, \quad \delta = \tan^{-1}\frac{P}{Q} - \beta_{rot},\\
&a = w_{orb}/w_{rot} \approx 1/365.26, \quad w_{orb}t = a\xi_{rot},\\
&n = \left(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta\right) \quad \xi_{rot} = w_{rot}t,
\end{aligned}\right\} \tag{2}$$

where $\theta, \phi, R_e, R_{se}, w_{orb} w_{rot}, \alpha$ and ε represent respectively the celestial co-latitude, longitude, Earth radius, average distance between Earth centre from the origin of SSB frame, orbital and rotational angular velocity of the Earth, co-latitude of the detector and obliquity of the ecliptic. Here β_{orb} and β_{rot} are the initial azimuth of the Earth and detector respectively.

To estimate N_{sky}, it is sufficient to consider either of the two polarisation of the signal given as,

$$h_+(t) = h_{o_+}\cos\left[\Phi(t)\right], \tag{3}$$

$$h_\times(t) = h_{o_\times}\sin\left[\Phi(t)\right], \tag{4}$$

hence, we consider the '+' polarisation of amplitude unity, given as

$$h(t) = \cos\left[\Phi(t)\right], \tag{5}$$

where $h_{o_+}, h_{o_\times}$ are constant amplitude of the two polarizations.

Now let us assume *h*(*t*) to be given on finite time interval $\left[0, T_{obs}\right]$ assumed to be the observation period. Now its straight forward to obtain FT in the similar way as obtain by Srivastava and Sahay (2002b), and may be given as

$$\begin{aligned}
\tilde{h}(f) = \int_0^{T_{obs}}\cos\left[\Phi(t)\right]\mathrm{e}^{-\mathrm{i}2\pi ft}\mathrm{d}t \simeq\\
\simeq \frac{\nu}{2w_{rot}}\sum_{k=-\infty}^{k=\infty}\sum_{m=-\infty}^{m=\infty}\mathrm{e}^{\mathrm{i}\mathcal{A}}\mathcal{B}\left[\tilde{\mathcal{C}} - \mathrm{i}\tilde{\mathcal{D}}\right];
\end{aligned} \tag{6}$$

where

$$\left.\begin{aligned}
&\nu = \frac{f_o - f}{f_{rot}}\\
&\mathcal{A} = \frac{(k+m)\pi}{2} - \mathcal{M}\\
&\mathcal{B} = \frac{J_k(Z)J_m(N)}{\nu^2 - (ak+m)^2}\\
&\tilde{C} = \sin\nu\xi_o \cos(ak\xi_o + m\xi_o - k\sigma - m\delta)\\
&\quad - \frac{ak+m}{\nu}\{\cos\nu\xi_o \sin(ak\xi_o + m\xi_o - k\sigma - m\delta) + \sin(k\sigma + m\delta)\}\\
&\tilde{\mathcal{D}} = \cos\nu\xi_o \cos(ak\xi_o + m\xi_o - k\sigma - m\delta)\\
&\quad + \frac{ak+m}{\nu}\{\sin\nu\xi_o \sin(ak\xi_o + m\xi_o - k\sigma - m\delta) - \cos(k\sigma + m\delta)\}\\
&\xi_o = w_{rot}T_{obs}
\end{aligned}\right\}, \qquad (7)$$

where J stands for the Bessel function of first kind. Using the symmetrical property of Bessel function we reduce the computation time appreciably by rewriting $\tilde{h}(f)$ as

$$\begin{aligned}
\tilde{h}(f) \simeq \frac{\nu}{w_{rot}}\Bigg[&\frac{J_o(\mathcal{Z})J_o(\mathcal{N})}{2\nu^2}\Big[\{\sin\mathcal{M} - \sin(\mathcal{M} - \nu\xi_o)\}+\\
&+\mathrm{i}\{\cos\mathcal{M} - \cos(\mathcal{M} - \nu\xi_o)\}\Big] + J_o(Z)\sum_{m=1}^{m=\infty}\frac{J_m(\mathcal{N})}{\nu^2 - m^2}\Big[(\mathcal{Y}\mathcal{U} - \mathcal{X}\mathcal{V}) -\\
&-\mathrm{i}(\mathcal{X}U + \mathcal{Y}\mathcal{V})\Big] + \sum_{k=1}^{k=\infty}\sum_{m=-\infty}^{m=\infty}\mathrm{e}^{i\mathcal{A}}\mathcal{B}(\tilde{C} - i\tilde{\mathcal{D}});
\end{aligned} \qquad (8)$$

$$\left.\begin{aligned}
&\mathcal{X} = \sin(\mathcal{M} - m\pi/2)\\
&\mathcal{Y} = \cos(\mathcal{M} - m\pi/2)\\
&\mathcal{U} = \sin\nu\xi_o \cos m(\xi_o - \delta) - \frac{m}{\nu}\{\cos\nu\xi_o \sin m(\xi_o - \delta) - \sin m\delta\}\\
&\mathcal{V} = \cos\nu\xi_o \cos m(\xi_o - \delta) + \frac{m}{\nu}\sin\nu\xi_o \sin m(\xi_o - \delta) - \cos m\delta
\end{aligned}\right\}. \qquad (9)$$

The transform contains double infinite series of Bessel function. However, for analysis the order of Bessel function required to compute $\tilde{h}(f)$ in the infinite series are given as (Srivastava and Sahay, 2002b).

$$k \approx 3133.22 \times 10^3 \sin\theta \left(\frac{f_o}{1\text{kHz}} \right), \tag{10}$$

$$m \approx 134 \left(\frac{f_o}{1\text{kHz}} \right). \tag{11}$$

The accuracy and range of validity for large order and/or argument has been discussed by Chishtie et al. (2005).

3. INDEPENDENT POINTS IN THE SKY

The study of the independent points for an all sky search has been made by many research workers [Schutz (1991), Brady et al. (1998), Brady and Creighton (2000), Jaranowski and Królak (1999, 2001), Astone et al. (2002)] for the coherent and/or incoherent search. The coherent search means cross correlating the data with the bank of search templates. While incoherent search implies adding of the power spectra by dividing the data into *N* subsets, performing a full search for each subset, and adds up the power spectra of the resulting searches. In this case, there is loss in *S*/*N* ratio by a factor of $\sqrt{N}$ in relation to coherent search as power spectra are added incoherently. However, irrespective of the method of search, the optimal spacing in the (θ, ϕ) parameters for an all sky search is a problem of interest.

For the coherent search, one have to make bank of search templates to detect the signal. The bank of search templates are discrete set of signals from among the continuum of possible signals. Consequently all the signals will not get detected with equal efficiency. However, it is possible to choose judiciously the set of templates so that all the signals of a given amplitude are detected with a given minimum detection loss. Fitting factor (*FF*) is one of the standard measure for deciding what class of wave form is good enough and quantitatively describes the closeness of the true signals to the template manifold in terms of the reduction of *S*/*N* arising due to the cross correlation of a signal outside the manifold with the best matching templates lying inside the manifold, given as

$$FF = \frac{\left\langle h(f) \middle| h_T(f;\theta_T,\phi_T) \right\rangle}{\sqrt{\left\langle h_T(f;\theta_T,\phi_T) \middle| h_T(f;\theta_T,\phi_T) \right\rangle \left\langle h(f) \middle| h(f) \right\rangle}}, \tag{12}$$

where *h*(*f*) and $h_T\left(f;\theta_T,\phi_T\right)$ represent the FTs of the actual signal and the templates respectively. The inner product of two signal h_1 and h_2 is defined as

$$\left\langle h_1 \middle| h_2 \right\rangle = 2\int_0^\infty \frac{\tilde{h}_1^*(f)\tilde{h}_2(f)+\tilde{h}_1(f)\tilde{h}_2^*(f)}{S_n(f)}\,\mathrm{d}f = \\ = 4\int_0^\infty \frac{\tilde{h}_1^*(f)\tilde{h}_2(f)}{S_n(f)}\,\mathrm{d}f \tag{13}$$

where * denotes complex conjugation and $S_n(f)$ is the spectral noise density of the detector.

To estimate optimal spacing in the parameters space one have to careful investigate the parameters contain in the phase of the modulated signal. Hence we check the effect for different β_{orb} in $\tilde{h}(f)$ for a week data set for the LIGO detector at Livingston [the position and orientation of the detector can be found in Allen (1995)] of unit amplitude signal for $f_o = 50$ Hz and $(\theta,\phi)=(\pi/18,\pi/4)$. We take the ranges of *k* and *m* as 1 to 27,300 and –15 to 15 respectively. We found that Earth azimuth affects the FM spectrum severely. Hence, for the coherent search, we investigate its effect for the number of independent of points in the sky.

To estimate N_{sky}, we consider the LIGO detector at Livingston, receive a PGW signal of frequency $f_o = 50$ Hz Hz from a source located at $(\theta,\phi)=(1^\circ,45^\circ)$. First, we chosen the data set such that $\beta_{orb}=0$ at $t = 0$. In this case we take the ranges of *k* and *m* as 1 to 2800 and –15 to 15 respectively and bandwidth equal to $50\pm 3.28\times 10^{-4}$ Hz for the integration. Now, we select the spacing $\Delta\theta = 0.45\times 10^{-4}$, thereafter we maximize over ϕ by introducing spacing $\Delta\phi$ in the so obtained N_{sky} and determine the resulting *FF*. In similar manner we obtain N_{sky} for $\beta_{orb}=\pi/6$ and $\pi/2$. The results obtained are shown in the Fig. 1. We also plot the templates spacing $\Delta\phi$ in the ϕ-parameter with FF shown in the Fig. 2. Interestingly, the nature of these curves are similar. We have obtained a best fit of the graphs, given as

$$N_{sky} = 10^{18}\left[a_\circ + a_1x - a_2x^2 + a_3x^3 - a_4x^4 + a_5x^5 - a_6x^6 + a_7x^7\right]; \\ 0.85 \le x \le 0.995; \tag{14}$$

$$FF = b_\circ + b_1y - b_2y^2 + b_3y^3;\quad 0.037 \le y \le 0.69; \tag{15}$$

where $a_\circ ... a_7$ and $b_\circ ... b_3$ are constants as given in Table 1 and 2 respectively.

Table 1

Coefficients of the best fit graphs obtained for the N_{sky} with *FF*

β_{orb}	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
0	–1.71829	13.1670	43.2234	78.7956	86.1505	56.4921	20.5716	3.20919
π/6	–35.7699	273.770	897.636	1634.42	1784.84	1168.99	425.178	66.2488
π/2	–40.7053	311.849	1023.49	1865.42	2039.12	1336.86	486.720	75.9142

Table 2

Coefficients of the best fit graphs obtained for the *FF* with $\Delta\phi$

β_{orb}	$b_\circ$	b_1	b_2	b_3
0	0.994123	0.149451	3.94494	2.64819
π/6	0.993544	2.00186	366.375	1651.88
π/2	0.996774	2.11099	2914.21	37817.3

In view of the above investigation, the spacing $\Delta\phi$ in the ϕ-parameter may be expressed as

$$\Delta\phi = G\left(FF, f_\circ, \theta, \phi, T_{obs}, \beta_{orb}\right). \quad (16)$$

Equation (15) is a third order polynomial, hence complicates the solution. However, one would like to do data analysis for *FF* > 0.90. Therefore from Fig. 2 we obtain a very good dependence of *FF* for minimum N_{sky} and may be given as

$$FF = -2.85657\Delta\phi^2 + 0.00230059\Delta\phi + 1.00018. \quad (17)$$

From Eqs. (16), and (17), we may write

$$\mathcal{F}\left(FF, 50, 1^\circ, 45^\circ, 1w, \beta_{orb}\right) \approx 4.02684\times10^{-4} \pm 0.591667\sqrt{1.00018 - FF}. \quad (18)$$

Above equation can be relevant to make trade-off between computational costs and sensitivities i.e. for the selected *FF* one can estimate N_{sky}. However, there is no unique choice for it. Here we are interested in the estimation of N_{sky} such that the spacing is maximum resulting into the least number of points require for the search. As mentioned earlier, there is stringent requirement on reducing computational costs. Accordingly, there is serious need of adopting some procedure/formalism to achieve this. For example, one may adopt the method of hierarchical search.

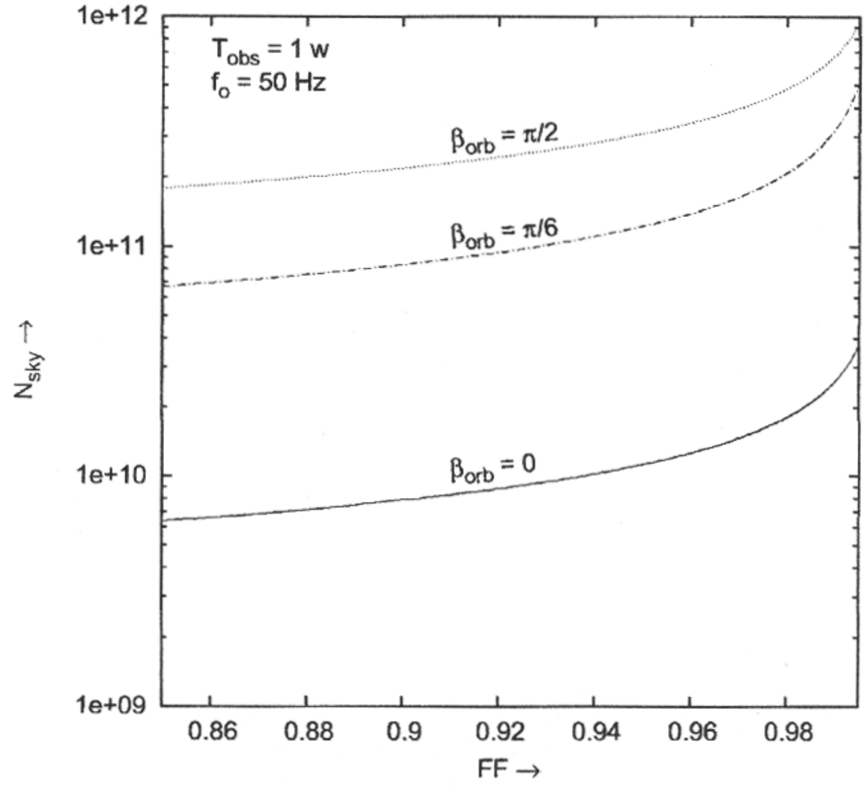

Fig. 1 – N_{sky} with FF at different β_{orb}.

4. COMPUTATIONAL COSTS

In view of the above analysis it will be interesting to know the feasibility of the all sky search with the target sensitivity of the advanced LIGO. The computational costs of the data analysis basically depends on the floating point operations (flops) require to perform the Fast Fourier transform (FFT). Hence in terms of FFT, the flops for the data reduction upto frequency f for T_{obs} of the interferometer output may be given as (Press et al., 1986)

$$N_{flops} = 2f\,T_{obs}\,\log_2\left(2f_{\max}T_{obs}\right). \tag{19}$$

Now for the given mismatch (FF), if N_p is the number of independent points to perform all sky search, then the flops will be

$$N_{flops}\left(FF, N_p\right) = 2f\,T_{obs}N_p\,\log_2\left(2f_{\max}T_{obs}\right). \tag{10}$$

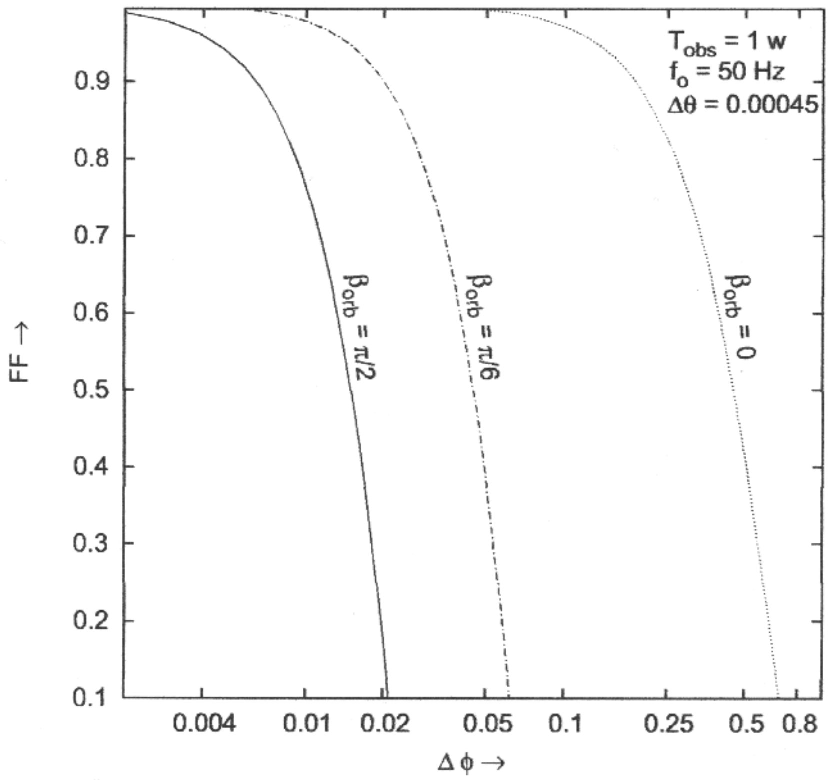

Fig. 2 –FF with $\Delta\phi$ at different β_{orb}.

Hence, for the mismatch of 3% and without manipulating the data in reference to β_{orb}, the flops for the search of PGW signal upto 50 Hz in the week data set will be 2.38×10^{19}, 2.58×10^{20} and 6.23×10^{20} for $\beta_{orb} = 0, \pi/6$ and $\pi/2$ respectively, assuming other operation need negligible flops compare to FFT. However, the lower cut off frequency of the LIGO I/II is 10/40 Hz. Hence the analysis shall be done above the lower cut off. Also, the search will be more significant if one perform in the most sensitive band of the detector. In this, if one would like to perform all sky search in a small band say 5 Hz then the minimum flops for the on-line analysis (a week data gets analyzed in a ~ week time) will be 3.44×10^{12}. The flops require may be further reduce, if one perform hierarchal search. Hence, it may be feasible to perform limited frequency all sky search of signal amplitude $\gtrsim 10^{-26}$ in the output of such a sensitive detector with a ~Tflops computer.

5. CONCLUSIONS

In this paper we have incorporated the initial azimuth of the Earth in the FT of the frequency modulated PGW signal and investigated its effect in the independent points in the sky require for the search of PGW. We found that the N_{sky} for the search in the output of one week data set varies significantly with β_{orb}. For the case investigated here, we observe that for *FF* = 0.97 approximately 1.53×10^{10}, 1.66×10^{11} and 4.0×10^{11} N_{sky} will be require when $\beta_{orb} = 0, \pi/6$ and $\pi/2$ respectively. Hence, the analysis may be useful to reduce the computational cost for a coherent all sky search. However, also the inspection of the phase of the modulated signal reveals that reduction in N_{sky} depends on time scale of integration, shorter the T_{obs} more the difference in N_{sky}.

The reduction in N_{sky} is large, so we studied the feasibility of all sky search in reference to advanced LIGO and found that in the band of 5 Hz one may perform on-line all sky search of the PGW signal of amplitude $\gtrsim 10^{-26}$ with a ~Tflops computer. The relation given by equation (18) may be useful to make trade-off between computational costs and sensitivities for the search of periodic gravitational waves. The issue to reduce the flops for the all sky search is a problem of interests and hence need more studies/investigations.

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