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Dedicated to Professor Ioan Gottlieb's
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BINET-TYPE EQUATIONS IN GENERAL RELATIVITY

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Abstract. It is shown that the geodesic trajectories in static charts with spherical symmetry of curved manifolds satisfy a relativistic versions of the Binet equation.

Key words: geodesic motion, central charts, Binet equation.

In general relativity, the geometric models help one to understand the relativistic kinematics and the compare this that of the well-known classical non-relativistic systems. In general, the geometric models are simple systems formed by given curved backgrounds where free test particles move along geodesic trajectories without to modify the geometry.

The relativistic correspondent of the non-relativistic classical central motion is the (classical) free motion of a test particle on spherically symmetric static backgrounds [1, 2, 3] whose line elements have the following general form:

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - \left[B(r)\delta_{ij} + C(r)x^i x^j\right]\mathrm{d}x^i\mathrm{d}x^j, \tag{1}$$

where A, B and C are arbitrary differentiable of r. Obviously this metric remains invariant under time translations and space rotations. Therefore, the energy as well as the angular momentum are conserved. These conversation laws can be easily written in Cartesian coordinates since these transform manifestly covariant under rotations. Thus, supposing that the test particle has the mass m using the Noether theorem in the (3+1) Hamiltonian formalism, we obtain the *energy* conservation.

$$\frac{E}{mc} = A\frac{\mathrm{d}t}{\mathrm{d}s} \tag{2}$$

and the conservation of the *angular momentum*,

$$L^{ij} = E\frac{B}{A}\left(x^i\dot{x}^j - x^j\dot{x}^i\right). \tag{3}$$

Furthermore, we introduce [4] three new functions of r, denoted by α, β and γ, so that

$$A = c^2 \frac{\alpha}{\beta}, \quad B = \frac{\alpha}{\beta\gamma}, \quad C = \frac{\alpha}{\beta^2\gamma} \frac{\gamma - \beta}{r^2}. \tag{4}$$

In addition, we take $L = L^{12} \neq 0$ and $L^{23} = L^{13} = 0$ in order to keep the trajectory on the shell $x^3 = 0$. The form of the geodesics of the central motion in spherical coordinates can be obtained directly in terms of the functions α, β and γ starting with the line element in spherical coordinates

$$\mathrm{d}s^2 = c^2 \frac{\alpha}{\beta} \mathrm{d}t^2 - \frac{\alpha}{\beta^2} \mathrm{d}r^2 - \frac{\alpha}{\beta\gamma} r^2 \left(\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\phi^2 \right). \tag{5}$$

Considering the conservation of the energy and of the angular momentum along to the third axis, we obtain the equations of motion on the space shell $\theta = \pi / 2$ [4],

$$\ddot{r} + w' = 0, \tag{6}$$

$$\dot{\phi} = \frac{\gamma}{r^2} \frac{Lc^2}{E}, \tag{7}$$

where

$$w(r) = \frac{c^2}{2} \left(\alpha \frac{m^2 c^4}{E^2} - \beta + \frac{\beta\gamma}{r^2} \frac{L^2 c^2}{E^2} \right) \tag{8}$$

is the *radial potential*. Notice that there exists the prime integral

$$\dot{r}^2 + c^2 \left(\alpha \frac{m^2 c^4}{E^2} - \beta + \frac{\beta\gamma}{r^2} \frac{L^2 c^2}{E^2} \right) = 0, \tag{9}$$

which allows us to directly integrate the radial equation (6).

Thus we obtain a new form of the geodesic equations of the relativistic central motion in Cartesian and spherical coordinates. Obviously, this approach has some technical advantages. One of them is that the potential w is linear in α and β so that in the non-relativistic limit the potential energy should have a term proportional with α–β. Moreover, we see that the angular relativistic effects could be produced only by the function γ.

Now we are able to derive the relativistic equivalent of the Binet equation since this can be easily written in terms of our new function α, β and γ. Let us first consider

$$Z = \frac{1}{r} \tag{10}$$

as a function of the angle ϕ. Then from Eqs. (7) and (8) we obtain the following relativistic Binet-type equation

$$\gamma^2\left[\frac{\mathrm{d}Z(\phi)}{\mathrm{d}\phi}\right]^2+\beta\gamma Z(\phi)^2+\frac{1}{L^2c^2}\left(\alpha m^2c^4-\beta E^2\right)=0, \qquad (11)$$

which given the form of the curved trajectories when $L\neq 0$. Of course, herein the functions α, β and γ have to be considered functions of Z instead of r. Notice that in the particular case of $L=0$ the trajectory is linear and the Binet-type equation does not make sense.

Hence, thank to our new functions α, β and γ, we obtained a simple form of the geodesic trajectories of any relativistic central motion.

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