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**OPTICS**

# THE PHOTON WAVE FUNCTION AND THE FRESNEL FORMULAS

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*Abstract*. Electromagnetic phenomena can be either classically described with the help of Maxwell's equations, or by the methods of quantum computing and Schrödinger equation. In the present paper we will describe the phenomena of reflection and refraction using only quantum mechanical methods. Finally, we get Fresnel's equations without to call on the classical electromagnetic equations. The method involves the use of wave functions of the photon in the momentum representation, compiled with a polarization term, and keeps the promise of a simple way of calculating the behavior of photonic devices as used in optoelectronics.

*Key words:* Schrödinger type equation, Fresnel formulas.

## 1. INTRODUCTION

As it is well known, the quantum mechanical calculation methods are very advanced as compared to those of classical physics ones. For this reason, describing a physical phenomenon by the methods of quantum mechanics is usually much simpler in comparison to the same phenomenon described by the methods of classical physics. In the following, we will get the Fresnel's equations using only the mathematics of quantum mechanics.

Generally, the wave-particle duality applied to electromagnetic waves implies that the lower the wavelength, that is the higher the photon energy, the more defined becomes the particle behavior, as is the case for X- and Γ-rays. The phenomenon of photon tunneling has been specified for the first time by Heisenberg [1] when referring to the passage of light through a thin foil of gold. In this seminal study the propagation of the electromagnetic wave through the thin gold foil has been assimilated to the photon penetration through a potential barrier. This concept was later used also by other authors [2-4] in order to describe the photon propagation through a dielectric using the mathematical device of quantum mechanics.

## 2. THE LINK BETWEEN THE PHOTON WAVE FUNCTION AND THE ELECTROMAGNETIC FIELDS WAVE

The method of the wave function of the photon is presented in [5–7] with the help of the hexa-vector $Q$ introduced by Plücker and Kayley [8, 9]. Thus, we will define

$$\vec{Q}_A = \vec{E} + \mathrm{i}\vec{B}, \qquad \vec{Q}_B = \vec{D} + \mathrm{i}\vec{H}, \tag{1}$$

with

$$\vec{D} = \varepsilon\vec{E}, \qquad \vec{B} = \mu\vec{H}. \tag{2}$$

The photon wave function is a linear combination of these two vectors

$$\vec{\Psi} = K_A\vec{Q}_A + K_B\vec{Q}_B, \tag{3}$$

where $K_A$ and $K_B$ are complex coefficients. For a homogeneous and transparent medium we have

$$\vec{\Psi} = |C|\frac{1}{\sqrt{8\pi}}\left(\frac{\sqrt{\varepsilon}}{n+1}\vec{Q}_A + \frac{\sqrt{\mu}}{n+1}\vec{Q}_B\right) = |C|\frac{1}{\sqrt{8\pi}}\left(\sqrt{\varepsilon}\vec{E} + \mathrm{i}\sqrt{\mu}\vec{H}\right), \tag{4}$$

where $|C|^2$ is the normalization factor. By identifying the constants in eq. (4) we obtain the following system of equations

$$K_A + \varepsilon K_B = |C|\frac{1}{\sqrt{8\pi}}\sqrt{\varepsilon}, \quad K_B + \mu K_B = |C|\frac{1}{\sqrt{8\pi}}\sqrt{\mu}, \tag{5}$$

with the solutions

$$K_A = |C|\frac{1}{\sqrt{8\pi}}\frac{\sqrt{\varepsilon}}{n+1}, \quad K_B = |C|\frac{1}{\sqrt{8\pi}}\frac{\sqrt{\mu}}{n+1}. \tag{6}$$

In this case the normalization condition reads

$$\int \vec{\Psi}^{+}\vec{\Psi}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = |C|^2\int\frac{\vec{E}\vec{D} + \vec{H}\vec{B}}{8\pi}\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{7}$$

The wave function from (4) can also be represented in the following matrix form

$$\vec{\Psi} \to |C|\frac{1}{\sqrt{8\pi}}\begin{bmatrix}\sqrt{\varepsilon}E_x + \sqrt{\mu}H_x\\ \sqrt{\varepsilon}E_y + \sqrt{\mu}H_y\\ \sqrt{\varepsilon}E_z + \sqrt{\mu}H_z\end{bmatrix} \equiv \begin{bmatrix}\Psi_x\\ \Psi_y\\ \Psi_z\end{bmatrix} \tag{8}$$

or using the algebra of spin 1

$$\vec{\delta}=\begin{bmatrix}0&0&0\\0&0&-\mathrm{i}\\0&\mathrm{i}&0\end{bmatrix}\vec{\imath}_1+\begin{bmatrix}0&0&\mathrm{i}\\0&0&0\\-\mathrm{i}&0&0\end{bmatrix}\vec{\imath}_2+\begin{bmatrix}0&-\mathrm{i}&0\\\mathrm{i}&0&0\\0&0&0\end{bmatrix}\vec{\imath}_3, \tag{9}$$

$$\vec{\delta}\times\vec{\delta}=\mathrm{i}\vec{\delta},\quad \vec{\delta}^{+}=\vec{\delta}, \tag{10}$$

the wave equation for photon becomes [7]

$$\frac{\partial}{\partial t}\left(\Psi^{+}\Psi\right)+\nabla\cdot\left(v\Psi^{+}\vec{S}\Psi\right)=0. \tag{11}$$

By adding and subtracting the term $\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\Psi$ we finally get

$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi+\frac{1}{c^2}\left(1-n^2\right)\frac{\partial^2\Psi}{\partial t^2}=0. \tag{12}$$

In the case of propagation through a dielectric medium eq. (12) can be written

$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi-K^2\Psi=0,$$
$$K\equiv\frac{\omega}{c}\sqrt{1-n^2}. \qquad 13)$$

With the notation introduced in [10.11] the potential of dielectric medium has the form

$$U=c\hbar K=\hbar\omega\sqrt{1-n^2}\;. \tag{14}$$

Finally, the energy of a quantum particle, in particular that of a photon ($m_0=0$) which moves in an electromagnetic field characterized by a potential vector $\vec{A}$ and a scalar one $\Phi$, as well as in our scalar potential $U$, associated to the dielectric medium, may be written:

$$E=q\Phi+\left[\left(m_0c^2+U\right)^2+\left(c\vec{p}-q\vec{A}\right)^2\right]^{\frac{1}{2}}. \tag{15}$$

## 3. PHOTONS AND STEP POTENTIAL

It should be emphasized that the photon wave function introduced above has not the significance of amplitude of spatial location like in non-relativistic quantum mechanics, the notion of photon coordinate being void of any physical content.

However, the concept proves to be beneficial and is used by some authors who apply quantum mechanics methods to describe the electromagnetic phenomena with the help of photons [12].

Let us further consider the step potential from Fig.1 associated to the interface between two dielectric media characterized by the potentials $U_1$ and $U_2$.

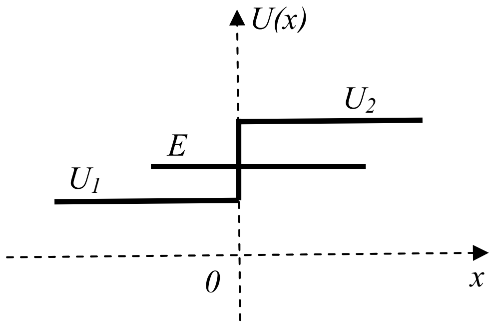


Fig. 1 – One-dimensional potential for along $U_2$ direction, perpendicular to dielectric interface.

The wave functions of the photon in momentum representation, solutions of Eq. (12), are in case of one-dimensional potential from Fig.1 the following:

$$x \leq 0, \quad \Phi_1 = A\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_{x1}x} + B\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p_{x1}x}, \tag{16}$$

$$x \geq 0, \quad \Phi_2 = C\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_{x2}x}. \tag{17}$$

In Fig. 1 we suppose that the incident particle, in our case the photon, moves from left to right. We have to note that the momenta $p_{x1}$ and $p_{x2}$ may be real or imaginary inasmuch as the quantity $\left(E^2 - U^2\right)$ may be positive or negative (see eq (15)), which means that we have a progressive or a real decreasing exponential wave. We mention that there are media with the refractive index $n < 1$, in which case the potential $U$ is imaginary, and the quantity $\left(E^2 - U^2\right)$ can become negative. In addition, we will also assume that the particle - photon possesses an energy-momentum quadrivector $E, \vec{p}$ and a unit vector, $\overline{\Pi}$, orthogonal to the moving direction of the particle, the latter being the photon polarization vector. Consequently, the wave function will be completed with this additional vector, that is the the new wave function is the product of the original wave function depending on coordinate $\Phi(r)$, and the polarization vector $\overline{\Pi}$, namely

$$\vec{\Xi} = \Phi(r)\vec{\Pi}\,. \tag{18}$$

The reflection and transmission coefficients on the step potential from Fig. 1 are obtained immediately from the continuity of the corresponding wave functions

$$x \le 0, \quad \vec{\Xi}_1 = A\vec{\Pi}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_{x1}x} + B\vec{\Pi}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p_{x1}x}, \tag{19}$$

$$x \ge 0, \quad \vec{\Xi}_2 = C\vec{\Pi}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_{x2}x}. \tag{20}$$

## 4. THE FRESNEL RELATIONS

The continuity conditions of (19) and (20) at the potential step in $x = 0$ leads to

$$\frac{\vec{\Pi}_1}{\vec{\Pi}_0}\frac{B}{A} = \frac{p_{x1} - p_{x2}}{p_{x1} + p_{x2}}, \tag{21}$$

$$\frac{\vec{\Pi}_2}{\vec{\Pi}_0}\frac{C}{A} = \frac{2p_{x1}}{p_{x1} + p_{x2}}. \tag{22}$$

Using further the relations (21) and (22) and the fact that the passage of particles from medium 1 characterized by the potential $U_1$ to medium 2 characterized by potential $U_2$ conserves the particle momentum parallel to the dielectric interface [12], a simple calculation (see Annex), leads immediately to

$$\frac{\vec{\Pi}_1}{\vec{\Pi}_0}\frac{B}{A} = -\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)}, \tag{23}$$

and, respectively, to

$$\frac{\vec{\Pi}_2}{\vec{\Pi}_0}\frac{C}{A} = -\frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1 + \theta_2)}. \tag{24}$$

We find two particular situations, any other case being a linear combination of these two, namely:

a) the polarization vector is perpendicular to the incidence plane $\vec{\Pi}_\perp$ ;

b) the polarization vector is parallel to the incidence plane $\vec{\Pi}_\parallel$ ;

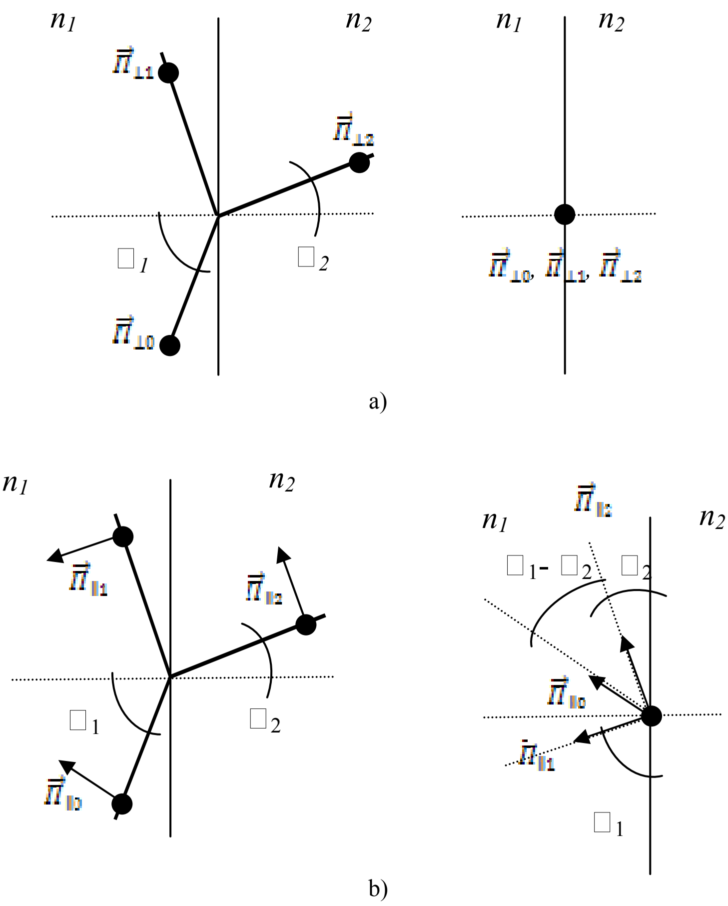


Fig. 2 – Indices 0, 1 and 2 refer to the incident, reflected and transmitted photon:
a) $\overrightarrow{\Pi_\perp}$ vectors normal to the incidence plane ; b) $\overrightarrow{\Pi_{\|}}$ vectors parallel to the incidence plane.

As can be seen from Fig. 2a the perpendicular polarization vectors $\overrightarrow{\Pi_\perp}$ are parallel to each other, and so the direction cosines are

$$\frac{\vec{\Pi}_{\perp 1}}{\vec{\Pi}_{\perp 0}} = 1, \tag{25}$$

$$\frac{\vec{\Pi}_{\perp 2}}{\vec{\Pi}_{\perp 0}} = 1, \tag{26}$$

This allows us to write for the amplitude of the wave functions, in conformity to Eq. (23) and (24) the following relationships

$$B_{\perp} = -A_{\perp} \frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)}, \tag{27}$$

$$C_{\perp} = A_{\perp} \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1 + \theta_2)}, \tag{28}$$

that is Fresnel relations for the case of perpendicular polarization.

In the case of parallel polarization, for Fig. 2b we get the polarization vectors ratios:

$$\frac{\vec{\Pi}_{\|2}}{\vec{\Pi}_{\|0}} = \cos(\theta_1 - \theta_2), \tag{29}$$

and

$$\frac{\vec{\Pi}_{\|2}}{\vec{\Pi}_{\|1}} = \cos\left[\pi - (\theta_1 + \theta_2)\right]. \tag{30}$$

Dividing further the last two relationships it results

$$\frac{\vec{\Pi}_{\|1}}{\vec{\Pi}_{\|0}} = -\frac{\cos(\theta_1 - \theta_2)}{\cos(\theta_1 + \theta_2)}. \tag{31}$$

Finally, using Eq. (29) and (31) in Eq. (23) and (24) we get for photons with parallel polarization vectors contained in the incident plane following amplitudes:

$$B_{\|} = A_{\|} \frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)} \frac{\cos(\theta_1 + \theta_2)}{\cos(\theta_1 - \theta_2)}, \tag{32}$$

$$C_{\|} = A_{\|} \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1 + \theta_2)} \frac{1}{\cos(\theta_1 - \theta_2)}, \tag{33}$$

or more

$$B_{\|} = A_{\|} \frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}, \tag{34}$$

$$C_{\|} = A_{\|} \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2)}. \tag{35}$$

The relationships (34) and (35) are Fresnel's relations for the case of parallel polarization.

We have obtained in this way the Fresnel relations, for perpendicular as well as for parallel polarization, without any appeal to Maxwell's equations, or to continuity relations for the electric and magnetic fields in the separation area between dielectrics.

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## ANNEX

To obtain eq. (23) and (24) we consider the following:

In order to obtain the equations (23) and (24) let us consider Fig.A1. When a particle crosses the interface between two media with potentials $U_1$ and $U_2$ respectively, the parallel momentum component to this interface is conserved [12], that is

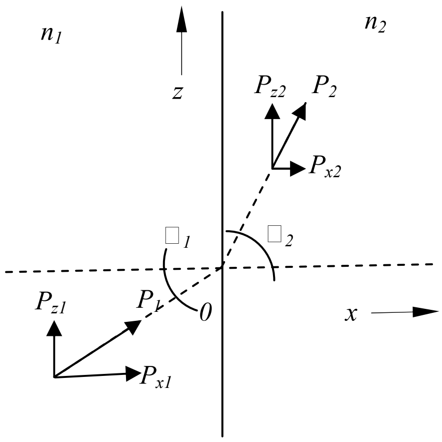


Fig. A1 – The photon moments in two dielectric media with $n_2 < n_1$.

$$p_{z1} = p_{z2}. \tag{A1}$$

Taking further into account that $p = p_0 n$, with $p_0$ the photon momentum in vacuum, and $n$ the refractive index of the medium, we can write in terms of the angles of Fig. A1 the light refraction law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{A2}$$

On the other hand, the normal momentum component to the interface between the two media is not conserved and is given by

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$$p_{x1,2} = p_{1,2} \cos\theta_{1,2}. \tag{A3}$$

or, in terms of the photon momentum in vacuum,

$$p_{x1,2} = \frac{p_{x1} - p_{x2}}{p_{x1} + p_{x2}} = \frac{p_0 n_1 \cos\theta_1 - p_0 n_2 \cos\theta_2}{p_0 n_1 \cos\theta_1 + p_0 n_2 \cos\theta_2} = \frac{\frac{n_1}{n_2}\cos\theta_1 - \cos\theta_2}{\frac{n_1}{n_2}\cos\theta_1 + \cos\theta_2}. \tag{A5}$$

and using (A2) we obtain eq. (23)

$$\frac{p_{x1} - p_{x2}}{p_{x1} + p_{x2}} = \frac{\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1 - \cos\theta_2}{\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1 + \cos\theta_2} = -\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)}. \tag{A6}$$

Similarly,

$$\frac{2p_{x1}}{p_{x1} + p_{x2}} = \frac{2p_0 n_1 \cos\theta_1}{p_0 n_1 \cos\theta_1 + p_0 n_2 \cos\theta_2} = \frac{2\frac{n_1}{n_2}\cos\theta_1}{\frac{n_1}{n_2}\cos\theta_1 + \cos\theta_2}, \tag{A7}$$

and using again (A2) we get eq. (24)

$$\frac{2p_{x1}}{p_{x1} + p_{x2}} = \frac{2\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1}{\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1 + \cos\theta_2} = \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1 + \theta_2)}. \tag{A8}$$