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IS THE RADIATION FIELD CALCULATION FROM JEFIMENKO'S EQUATIONS A NEW INSIGHT IN THE THEORY ?*

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Abstract. Motivated by a recent publication on the topic, we discuss some features of the multipolar expansion of the power radiated by a confined system of charges and currents. We investigate if the employment of Jefimenko's equations brings a new insight for the calculation of the radiation field. We show that the affirmation would be valid if one found an interesting example in which inverting the order between spatial derivatives and integration is not allowed. This is not usually the case in most of the calculations done in the radiation theory.

Key words: Jefimenko's equations, radiation field calculation.

1. INTRODUCTION

The electromagnetic field theory is considered a successful and well-defined theory. However, there are still several topics opened to new theoretical and pedagogical contributions. One of them is related to the importance of Jefimenko's equations for expressing the electric and magnetic field when discussing the radiation theory. The main aim of the present paper is to investigate if the use of Jefimenko's equations brings, as sometimes presented in the literature, a new insight in the calculation of the radiation field. Since it is directly related, we also discuss an advantageous procedure for the multipolar expansion in Cartesian coordinates.

A relatively recent textbook [1] and a paper [2], the last related to the importance of Jefimenko's equations for expressing the electric and magnetic field when discussing the radiation theory, brought our attention on a very hard formalism employed for the calculation of even the first three or four terms of the

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expansion series of the electromagnetic field. The traditional multipole expansion of the electromagnetic field in Cartesian coordinates is exposed in electrodynamics textbooks, as the well-known Refs. [3] and [4]. Ordinarily, these expansions are calculated only in the first two or three orders, the higher-orders being considered too complicated. Usually, an alternative treatment based on the spherical tensors and the solutions of Helmholtz equation is preferred. Though there are some prescriptions in the literature [5, 6, 7] for calculating higher-order terms of the multipole series based on a simple algebraic formalism of tensorial analysis, it seems that there is some reticence in using this last technique. For this reason, in the present paper we also show how one can hide, as much as possible, the higher-order tensors behind some vectors, reducing the calculation technique to the formalism of an ordinary vectorial algebra or analysis. Based on this vectorial/tensorial algebra technique, we investigate several alternatives for the calculation of the radiation field. We show that unless one finds an example where the spatial derivative and the integral operations can not be inverted, the use of Jefimenko's equations it is an unnecessary complication.

We start in section 2 by shortly presenting the notation convention we use and by giving a general formalism for handling multipolar expansions in Cartesian coordinates. In section 3, we derive the radiated electric and magnetic field without using the retarded potentials, while in section 4 we present characteristics of the calculation for the radiation field when employing Jefimenko's equations. The advantages and disadvantages of different approaches are analyzed. The conclusions are presented in the last section.

2. MULTIPOLAR EXPANSION OF FIELD

We write Maxwell's equations with a notation independent of the unit system ("system free" Maxwell's equations):

$$\begin{aligned} &\nabla \times \boldsymbol{B} = \frac{\mu_0}{\alpha}\left(\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right), \quad \nabla \times \boldsymbol{E} = -\frac{1}{\alpha}\frac{\partial \boldsymbol{B}}{\partial t}, \\ &\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0}\rho, \end{aligned} \tag{1}$$

where $\varepsilon_0, \mu_0, \alpha$ are dimensional factors depending on the system of units and are satisfying the equation $\frac{\alpha^2}{\varepsilon_0 \mu_0} = c^2$. c is the vacuum light speed. Maxwell equations written in SI units are obtained from equations (1) for $\alpha = 1$ and the SI values of ε_0, μ_0. For the Gauss system of units, $\alpha = c,\ \varepsilon_0 = 1/4\pi,\ \mu_0 = 4\pi$.

With this notation, Jefimenko's equations can be written as [4]

$$\boldsymbol{B}(\boldsymbol{r},t)=\frac{\mu_0}{4\pi\alpha}\int_{\mathcal{D}}\nabla\times\frac{[\boldsymbol{J}]}{R}\mathrm{d}^3x', \tag{2}$$

$$\boldsymbol{E}(\boldsymbol{r},t)=-\frac{1}{4\pi\varepsilon_0}\int_{\mathcal{D}}\nabla\frac{[\rho]}{R}\mathrm{d}^3x'-\frac{\mu_0}{4\pi\alpha^2}\int_{\mathcal{D}}\frac{[\dot{\boldsymbol{J}}]}{R}\mathrm{d}^3x', \tag{3}$$

where $\boldsymbol{R}=\boldsymbol{r}-\boldsymbol{r}'$, $[f]=f(\boldsymbol{r}',t-R/c)$ and the origin O of Cartesian coordinates is in the domain $\mathcal{D}$. The support of charge and current distribution is supposed included in $\mathcal{D}$. If in equations (2) and (3) the order of the derivative and the integral is inverted, one obtains the well-known relations between fields and potentials:

$$\boldsymbol{B}=\nabla\times\boldsymbol{A},\ \boldsymbol{E}=-\nabla\Phi-\frac{1}{\alpha}\frac{\partial\boldsymbol{A}}{\partial t}, \tag{4}$$

with the retarded potentials

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{\mu_0}{4\pi\alpha}\int_{\mathcal{D}}\frac{[\boldsymbol{J}]}{R}\mathrm{d}^3x',\ \Phi(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\int_{\mathcal{D}}\frac{[\rho]}{R}\mathrm{d}^3x'. \tag{5}$$

We underline that equations (2) and (3) are rightfully called *Jefimenko's equations* when they are considered as a direct result of Maxwell's equations. Hence, their status of fundamental equations of the electromagnetic theory is conditioned and it may become valid only if the inversion of operation in equation (4) is not allowed. In [2], the authors derive the multipole expansion of the radiation field from equations (2) and (3) claiming to give an original demonstration specific for Jefimenko's equations, without employing the retarded potentials. We should agree with this claim if at least some calculation of the authors is different from those employing the potential multipole expansions which are generally used in literature. In the following, we search for a difference between the calculation presented in Ref. [2] and the standard one making use of potentials. The goal of the exposition below is to inform on some results regarding multipolar expansion in Cartesian coordinates, too.

Let us derive the multipolar expansion of the field $\boldsymbol{B}$ given by equation (2). Writing the integral as

$$\int_{\mathcal{D}}\partial_j\frac{[\boldsymbol{J}_k]}{R}\mathrm{d}^3x'=\int_{\mathcal{D}}\partial_j\left(\frac{J_k\left(\xi,t-\dfrac{R}{c}\right)}{R}\right)_{\xi=\mathbf{r}'}\mathrm{d}^3x', \tag{6}$$

we obtain the multipolar expansion of the magnetic field about O as function of $\boldsymbol{r}'$ using the Taylor series of the integrand:

$$\partial_j \frac{J_k\left(\xi, t-\frac{R}{c}\right)}{R} = \sum_{n\geq 0} \frac{(-1)^n}{n!} x'_{i_1} \ldots x'_{i_n} \partial_{i_i} \ldots \partial_{i_n} \partial_j \left(\frac{1}{r} J_k\left(\xi, t-\frac{r}{c}\right)\right).$$

Equation (2) can now be expressed as

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi\alpha} \boldsymbol{e}_i \, \varepsilon_{ijk} \int_{\mathcal{D}} \sum_{n\geq 0} \frac{(-1)^n}{n!} x'_{i_1} \ldots x'_{i_n} \partial_j \partial_{i_1} \ldots \partial_{i_n} \left(\frac{1}{r} [J_k]_0\right) \mathrm{d}^3 x'. \tag{7}$$

$\boldsymbol{e}_i$ are the unit vectors of the Cartesian axes and we employed the notation $[f]_0 = f(\boldsymbol{r}', t - r/c)$.

We assume one is allowed to invert orders of operations in equation (7) and to write:

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi\alpha} \boldsymbol{e}_i \, \varepsilon_{ijk} \sum_{n\geq 0} \frac{(-1)^n}{n!} \partial_j \partial_{i_1} \ldots \partial_{i_n} \left(\frac{1}{r} \int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_n} [J_k]_0\right) \mathrm{d}^3 x'. \tag{8}$$

Equation (8) represents the curl of the multipolar expansion of the vector potential $\boldsymbol{A}$. Thus one can perform firstly the multipolar expansion of this potential. It is the usual procedure.

No matter what procedure is employed, a constant in the calculation is the presence of a vector $\boldsymbol{a}(\boldsymbol{r},t;\zeta,n)$ defined by the Cartesian components:

$$a_k(\boldsymbol{r},t;\zeta,n) = \zeta_{i_1} \ldots \zeta_{i_n} \left(\frac{1}{r} \int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_n} J_k(\boldsymbol{r}',t) \, \mathrm{d}^3 x'\right). \tag{9}$$

Here, ζ can be either an operator or a number. Generalizing to the dynamic case an algorithm used in [8] for the magnetostatic field, we introduce in equation (9) the consequence of the continuity equation, written for $t_0 = t - r/c$:

$$J_k(\boldsymbol{r}',t_0) = \nabla'\left(x'_k \boldsymbol{J}(\boldsymbol{r}',t_0)\right) + x'_k \dot{\rho}(\boldsymbol{r}',t_0).$$

We obtain

$$a_k(\boldsymbol{r},t_0;\zeta,n) = \zeta_{i_1} \ldots \zeta_{i_n} \frac{1}{r} \left(\int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_n} \nabla'\left(x'_k [\boldsymbol{J}]_0\right) \mathrm{d}^3 x' + \dot{\mathrm{P}}_{i_1 \ldots i_n k}(t_0) \right), \tag{10}$$

where the Cartesian components of the $n^{\text{-th}}$ electric moment of the given charge distribution:

$$\mathrm{P}_{i_1 \ldots i_n}(t) = \int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_n} \rho(\boldsymbol{r}' t) \, \mathrm{d}^3 x' \tag{11}$$

are introduced. Let us define, for simplifying the notation, the vector

$$\boldsymbol{\mathcal{P}}\left(r,t;\zeta,n\right)=\boldsymbol{e}_k\zeta_{i_1}\dots\zeta_{i_{n-1}}\frac{\mathbf{P}_{i_1\dots i_{n-1}k}\left(t\right)}{r}. \tag{12}$$

Performing partial integration and taking into account that $\boldsymbol{J}$ vanishes on the surface $\partial\mathcal{D}$, equation (10) can be written and processed as follows:

$$\begin{aligned}
&a_k\left(\boldsymbol{r},t_0;\zeta,n\right)-\dot{\mathcal{P}}_k\left(\boldsymbol{r},t;\zeta,n+1\right)=-\zeta_{i_1}\dots\zeta_{i_n}\frac{1}{r}\int_{\mathcal{D}}x'_k\left[\boldsymbol{J}\right]_0\cdot\\
&\cdot\nabla'\left(x'_{i_1}\dots x'_{i_n}\right)\mathrm{d}^3x'=-n\zeta_{i_1}\dots\zeta_{i_n}\frac{1}{r}\int_{\mathcal{D}}x'_{i_1}\dots x'_{i_{n-1}}x'_k\left[\boldsymbol{J}_{\boldsymbol{i_n}}\right]_0\mathrm{d}^3x'=\\
&=-n\varepsilon_{ki_nq}\zeta_{i_n}\zeta_{i_1}\dots\zeta_{i_{n-1}}\frac{1}{r}\int_{\mathcal{D}}x'_{i_1}\dots x'_{i_{n-1}}\left(r'\times\left[\boldsymbol{J}\right]_0\right)_q\mathrm{d}^3x'-na_k\left(\boldsymbol{r},t_0;\zeta,n\right).
\end{aligned} \tag{13}$$

Introducing the $n^{\text{-th}}$ order magnetic moment, as in Ref. [8], by its Cartesian components

$$\mathbf{M}_{i_1\dots i_n}\left(t\right)=\frac{n}{\left(n+1\right)\alpha}\int_{\mathcal{D}}x'_{i_1}\dots x'_{i_{n-1}}\left(\boldsymbol{r}'\times\boldsymbol{J}\left(\boldsymbol{r}',t\right)\right)_{i_n}\mathrm{d}^3x', \tag{14}$$

and similarly to equation (12), the vector

$$\boldsymbol{\mathcal{M}}\left(\boldsymbol{r},t;\zeta,n\right)=\boldsymbol{e}_k\zeta_{i_1}\dots\zeta_{i_{n-1}}\frac{\mathbf{M}_{i_1\dots i_{n-1}k}\left(t\right)}{r}, \tag{15}$$

equation (13) becomes

$$\boldsymbol{a}\left(\boldsymbol{r},t_0;\zeta,n\right)=-\alpha\zeta\times\boldsymbol{\mathcal{M}}\left(\boldsymbol{r},t_0;\zeta,n\right)+\frac{1}{n+1}\dot{\boldsymbol{\mathcal{P}}}\left(\boldsymbol{r},t_0;\zeta,n+1\right). \tag{16}$$

With this result, the magnetic field from equation (8) can be expressed with the help of the vectors $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{P}}$:

$$\begin{aligned}
&\boldsymbol{B}\left(\boldsymbol{r},t\right)=\frac{\mu_0}{4\pi}\sum_{n\geq1}\frac{\left(-1\right)^{n-1}}{n!}\nabla\times\left(\nabla\times\boldsymbol{\mathcal{M}}\left(\boldsymbol{r},t_0;\nabla,n\right)\right)+\\
&+\frac{\mu_0}{4\pi\alpha}\sum_{n\geq0}\frac{\left(-1\right)^{n}}{\left(n+1\right)!}\nabla\times\dot{\boldsymbol{\mathcal{P}}}\left(\boldsymbol{r},t_0;\nabla,n+1\right)=\\
&=\nabla\times\frac{\mu_0}{4\pi}\sum_{n\geq1}\frac{\left(-1\right)^{n-1}}{n!}\left(\nabla\times\boldsymbol{\mathcal{M}}\left(\boldsymbol{r},t_0;\nabla,n\right)+\frac{1}{\alpha}\dot{\boldsymbol{\mathcal{P}}}\left(\boldsymbol{r},t_0;\nabla,n\right)\right).
\end{aligned} \tag{17}$$

From the last expression one has no problem in identifying the multipolar expansion of the vector potential $\boldsymbol{A}$:

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{\mu_0}{4\pi}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n!}\left(\nabla\times\boldsymbol{\mathcal{M}}(\boldsymbol{r},t_0;\nabla,n)+\frac{1}{\alpha}\dot{\boldsymbol{\mathcal{P}}}(\boldsymbol{r},t_0;\nabla,n)\right). \tag{18}$$

The calculation for the electric field can be performed in a similar manner. One obtains:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t)=&-\frac{\mu_0}{4\pi\varepsilon_0}\sum_{n\geq 0}\frac{(-1)^n}{n!}\nabla\left(\nabla\cdot\boldsymbol{\mathcal{P}}(\boldsymbol{r},t_0;\nabla,n)\right)-\\&-\frac{\mu_0}{4\pi\alpha}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n!}\left(\nabla\times\boldsymbol{\mathcal{M}}(\boldsymbol{r},t_0;\nabla,n)+\frac{1}{\alpha}\ddot{\boldsymbol{\mathcal{P}}}(\boldsymbol{r},t_0;\nabla,n)\right).\end{aligned} \tag{19}$$

Comparing equations (4) with (18) and (19), we single out the multipole expansion of the potential Φ:

$$\Phi(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\sum_{n\geq 0}\frac{(-1)^n}{n!}\nabla\cdot\boldsymbol{\mathcal{P}}(\boldsymbol{r},t_0;\nabla,n). \tag{20}$$

3. RADIATION FIELD

For calculating the radiation field it is sufficient retaining only terms of order $1/r$ and $1/r^2$ for $r\to\infty$. In most textbooks one retains only the terms of order $1/r$, the goal being, usually, only the derivation of the radiated energy or of the linear momentum. Actually, when the goal is the complete definition of the radiation field, one must be able to derive all transferring properties, including the angular momentum loss. These are, in fact, minimal conditions for defining a physical system. In the last case, the terms of order $1/r^2$ are also necessary (see Ref. [3] -Problem 2 at the end of Section 72, and also Refs. [9, 10]). Although the aim of the present paper is different, we also give the formula for introducing terms of order $1/r^2$ required for the evaluation of the angular momentum loss. The terms of the orders $1/r$ and $1/r^2$ are selected making use of formula [10]:

$$\begin{aligned}\partial_{i_1}\ldots\partial_{i_n}\left(\frac{f(t_0)}{r}\right)=&\frac{1}{r}\frac{(-1)^n}{c^n}\nu_{i_1}\ldots\nu_{i_n}\frac{\partial^n f(t_0)}{\partial t^n}+\\&+\frac{(-1)^n}{c^{n-1}r^2}\left(D_n\nu_{i_1}\ldots\nu_{i_n}-\nu_{\{i_1\ldots i_{n-2}}\delta_{i_{n-1}i_n\}}\right)\frac{\partial^{n-1}f(t_0)}{\partial t^{n-1}}.\end{aligned} \tag{21}$$

Again $t_0 = t - r/c$ and $\nu_i = x_i / r$. By $A_{\{i_1 \ldots i_n\}}$ we understand the sum over all the permutations of the symbols i_q that give distinct terms. The coefficients D_n are defined by the recurrence relations: $D_n = D_{n-1} + n, \quad D_0 = 0$. The formula from equation (21) can be easily proven by recurrence.

Considering equations (12) and (15), one can see that $\mathcal{P}(\boldsymbol{r}, t_0; \zeta, n)$ and $\mathcal{M}(\boldsymbol{r}, t_0; \zeta, n)$ are solutions of the homogeneous wave equation for $r \neq 0$. Consequently, one can apply the formula (21) for these quantities. Let us consider the multiple derivative as, for example,

$$\begin{aligned} &\partial_{j_1} \ldots \partial_{j_m} \mathcal{M}(\boldsymbol{r}, t_0; \nabla, n) = \\ &= \frac{(-1)^{n+m-1}}{c^{n+m-1}} \nu_{j_1} \ldots \nu_{j_m} \frac{\partial^{n+m-1} f(t_0)}{\partial t^{n+m-1}} \mathcal{M}(\boldsymbol{r}, t_0; \nu, n) + \mathcal{O}\left(\frac{1}{r^2}\right) \end{aligned} \tag{22}$$

and similarly for $\mathcal{P}(\boldsymbol{r}, t_0; \nabla, n)$. Using equation (22) and the equivalent relation for $\mathcal{P}(\boldsymbol{r}, t_0; \nabla, n)$ in equation (17), we obtain the first approximation of the multipolar expansion of radiated magnetic field, which is sufficient for calculating the radiated energy and the linear momentum:

$$\begin{aligned} \boldsymbol{B}_{rad}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{1}{n! c^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} \Big((\nu \cdot \mathcal{M}(\boldsymbol{r}, t_0; \nu, n)) \nu - \mathcal{M}(\boldsymbol{r}, t_0; \nu, n) - \\ - \frac{c}{\alpha} \nu \times \mathcal{P}(\boldsymbol{r}, t_0; \nu, n) \Big). \end{aligned} \tag{23}$$

For the field $\boldsymbol{E}_{rad}$, we obtain

$$\begin{aligned} \boldsymbol{E}_{rad}(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 1} \frac{1}{n! c^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} \Big(\nu \cdot \mathcal{P}(\boldsymbol{r}, t_0; \nu, n) \nu - \mathcal{P}(\boldsymbol{r}, t_0; \nu, n) + \\ + \frac{\alpha}{c} \nu \times \mathcal{M}(\boldsymbol{r}, t_0; \nu, n) \Big). \end{aligned} \tag{24}$$

We made use of the relation between $\varepsilon_0, \mu_0, \alpha$ and c. From equations (18) and (20), using equation (21), we get easily the expansions of the radiation field potentials:

$$\boldsymbol{A}_{rad}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{1}{n! c^n} \frac{\partial^n}{\partial t^n} \left(\mathcal{M}(\boldsymbol{r}, t_0; \nu, n) \times \nu + \frac{c}{\alpha} \mathcal{P}(\boldsymbol{r}, t_0; \nu, n) \right) \tag{25}$$

$$\Phi_{rad}(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 1} \frac{1}{n! c^n} \frac{\partial^n}{\partial t^n} \nu \cdot \mathcal{P}(\boldsymbol{r}, t_0; \nu, n). \tag{26}$$

Given the above expressions, one can verify that the relations between fields and potentials are:

$$\boldsymbol{B}_{rad} = \frac{1}{c}\left(\dot{\boldsymbol{A}}_{rad} \times \nu\right), \quad \boldsymbol{E}_{rad} = \frac{1}{\alpha}\left(\dot{\boldsymbol{A}}_{rad} \times \nu\right) \times \nu = \frac{c}{\alpha}\boldsymbol{B}_{rad} \times \nu. \tag{27}$$

These parts (proportional to $1/r$) from the radiated electric and magnetic fields are purely transverse fields, satisfying the properties (see also [10]):

$$\nu \cdot \boldsymbol{E}_{rad} = 0, \quad \nu \cdot \boldsymbol{B}_{rad} = 0, \quad \varepsilon_0 \left|\boldsymbol{E}_{rad}\right|^2 = \frac{1}{\mu_0}\left|\boldsymbol{B}_{rad}\right|^2. \tag{28}$$

4. IMPORTANCE OF JEFIMENKO'S EQUATIONS IN RADIATION THEORY

Similar to Ref. [3], for calculating the radiation field (in the first approximation), we approximate the retarded potentials by:

$$\begin{aligned} \Phi(\boldsymbol{r},t) &\approx \frac{1}{4\pi\varepsilon_0 r}\int_{\mathcal{D}} \rho\left(\boldsymbol{r}', t_0 + \frac{1}{c}\nu \cdot \boldsymbol{r}'\right) \mathrm{d}^3x', \\ \boldsymbol{A}(\boldsymbol{r},t) &\approx \frac{\mu_0}{4\pi\alpha r}\int_{\mathcal{D}} \boldsymbol{J}\left(\boldsymbol{r}', t_0 + \frac{1}{c}\nu \cdot \boldsymbol{r}'\right) \mathrm{d}^3x'. \end{aligned} \tag{29}$$

Introducing these expressions of the radiated potentials and retaining only the terms of order $1/r$, one obtains the relations from equation (27). In this calculation, the derivative operators must be introduced in the integral and so, the proof is indeed realized directly for the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. For $\boldsymbol{B}$, using the relations

$$\nabla \times \frac{\boldsymbol{J}\left(\boldsymbol{r}', t_0 + \dfrac{\nu \cdot \boldsymbol{r}'}{c}\right)}{r} = \left(\nabla \frac{1}{r}\right) \times \boldsymbol{J} + \frac{1}{r}\nabla\left(t_0 + \frac{\nu \cdot \boldsymbol{r}'}{c}\right) \times \dot{\boldsymbol{J}}$$

and

$$\nabla \frac{1}{r} = \mathcal{O}\left(\frac{1}{r^2}\right), \quad \nabla\left(t_0 + \frac{1}{c}\nu \cdot \boldsymbol{r}'\right) = -\frac{1}{c}\nu + \mathcal{O}\left(\frac{1}{r}\right), \tag{30}$$

we can write

$$\tilde{\boldsymbol{B}}(\boldsymbol{r},t) \approx -\frac{\mu_0}{4\pi\alpha c}\frac{1}{r}\nu \times \int_{\mathcal{D}} \dot{\boldsymbol{J}}\left(\boldsymbol{r}', t_0 + \frac{\nu \cdot \boldsymbol{r}'}{c}\right) \mathrm{d}^3x' + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{31}$$

The term of order $1/r$ represents the radiated field (precisely, the first approximation of this field). It is, in fact, equation (28) from Ref. [2] and we can consider, by examining Ref. [3], that this result was obtained long time ago from the expression (2) of the field B (see Ref. [11] and [12]). Look also in Ref. [2] to see the usage of Jefimenko's equation for calculating $\boldsymbol{B}_{\mathrm{rad}}$.

Introducing also the approximate expression $\tilde{\boldsymbol{E}}$ starting from equation (3), we obtain

$$\tilde{\boldsymbol{E}}(\boldsymbol{r},t) \approx \frac{\mu_0}{4\pi\alpha^2 r}\int_{\mathcal{D}}\left(c\dot{\rho}\left(\boldsymbol{r}',t_0+\frac{\nu\cdot\boldsymbol{r}'}{c}\right)\nu-\dot{\boldsymbol{J}}\left(\boldsymbol{r}',t_0+\frac{\nu\cdot\boldsymbol{r}'}{c}\right)\right)+\mathcal{O}\left(\frac{1}{r^2}\right). \quad (32)$$

Using the continuity equation written in the point **r'** at the retarded time $t-h/c$

$$[\dot{\rho}] = -\left[\nabla'\cdot\boldsymbol{J}(\boldsymbol{r}',t')\right]_{t'=t-R/c} = -\nabla'\cdot[\boldsymbol{J}]+\left[\dot{\boldsymbol{J}}\right]\cdot\nabla'(t-R/c), \quad (33)$$

it results

$$\dot{\rho}\left(\boldsymbol{r}',t_0+\frac{\nu\cdot\boldsymbol{r}'}{c}\right) = -\nabla'\cdot\boldsymbol{J}\left(\boldsymbol{r}',t_0+\frac{\nu\cdot\boldsymbol{r}'}{c}\right)+\frac{\nu}{c}\cdot\dot{\boldsymbol{J}}\left(\boldsymbol{r}',t_0+\frac{\nu\cdot\boldsymbol{r}'}{c}\right)+\mathcal{O}\left(\frac{1}{r}\right). \quad (34)$$

The first term from the right-hand side of the last equation gives no contribution to the integral from equation (32) and, after a simple algebraic calculation, we obtain the expression of $\boldsymbol{E}_{rad}$ from equation (27) (see also equations (66.3) from Ref. [3] and equation (30) from Ref. [2]).

We point out that equation (33) or, generally, the relation between the space derivative of a retarded quantity and the retarded value of the space derivative of the same quantity, should be well-known for each student from a class of electrodynamics since when writing the retarded potentials as solutions of the wave equation, it is necessary to verify the Lorenz condition. The verification can be realized in a direct calculation, a good exercise for a student. Only in this way one can be convinced that the retarded solutions are indeed electromagnetic potentials. For this goal, equation (33) is indispensable since the Lorenz condition appears as a consequence of the continuity equation.

Based on the above results, we can quickly obtain the multipole expansion of the radiation field. Considering the adequate Taylor series for the integrand in equation (31), we write

$$\begin{aligned}\boldsymbol{B}_{rad}(\boldsymbol{r},t) &= -\frac{\mu_0}{4\pi\alpha c}\frac{1}{r}\nu\times\int_{\mathcal{D}}\left(\dot{\boldsymbol{J}}\left(\xi,t_0+\frac{1}{c}\nu\cdot\boldsymbol{r}'\right)\right)_{\xi=r'}\mathrm{d}^3x' = \\ &= -\frac{\mu_0}{4\pi\alpha}\sum_{n\geq 0}\frac{1}{n!c^{n+1}}\frac{\partial^{n+1}}{\partial t^{n+1}}\nu\times\boldsymbol{a}(\boldsymbol{r},t_0;\nu,n).\end{aligned} \quad (35)$$

The vector $\boldsymbol{a}$ was defined in equation (9). After introducing its expression as a function of the vectors associated to the electric and magnetic moment, equation (16), the radiated magnetic field becomes

$$\begin{aligned}\boldsymbol{B}_{rad}\left(\boldsymbol{r},t\right)=&\frac{\mu_0}{4\pi}\sum_{n\geq1}\frac{1}{n!c^{n+1}}\frac{\partial^{n+1}}{\partial t^{n+1}}\nu\times\left(\nu\times\boldsymbol{\mathcal{M}}\left(\boldsymbol{r},t_0;\nu,n\right)\right)-\\&-\frac{\mu_0}{4\pi\alpha}\sum_{n\geq0}\frac{1}{\left(n+1\right)!c^{n+1}}\frac{\partial^{n+2}}{\partial t^{n+2}}\nu\times\boldsymbol{\mathcal{P}}\left(\boldsymbol{r},t_0;\nu,n+1\right).\end{aligned}\tag{36}$$

The final step is changing $n\rightarrow n-1$ in the second sum of equation (36). We obtain again equation (23).

We can say that the result (36), which is the multipole expansion of the radiation field, was determined via Jefimenko's equations.

Alternatively, after reaching the expression given by equation (7), we can perform the multipolar expansion of the radiation field in such a manner that we can also say that it is obtained via Jefimenko's equations. We extract from the integrand in equation (7) the terms of the order $1/r$ (and $1/r^2$ for a complete definition of this field):

$$\begin{aligned}\boldsymbol{B}_{rad}\left(\boldsymbol{r},t\right)&=-\frac{\mu_0}{4\pi\alpha}\boldsymbol{e}_i\varepsilon_{ijk}\nu_j\nu_{i_1}\ldots\nu_{i_n}\frac{1}{r}\int_{\mathcal{D}}\sum_{n\geq0}\frac{1}{n!c^{n+1}}x'_{i_1}\ldots x'_{i_n}\frac{\partial^{n+1}}{\partial t^{n+1}}\left[J_k\right]_0\mathrm{d}^3x'\\&=-\frac{\mu_0}{4\pi\alpha}\boldsymbol{e}_i\varepsilon_{ijk}\sum_{n\geq0}\frac{1}{n!c^{n+1}}\nu_j\frac{\partial^{n+1}}{\partial t^{n+1}}a_k\left(\boldsymbol{r}',t_0;\nu,n\right),\end{aligned}\tag{37}$$

arriving, as expected, to equation (35).

We remind the reader that for obtaining the results of equations (17) and (19) we admitted the commutation of the derivative with respect to the spatial coordinates, with the series expansion and the integral on the domain $\mathcal{D}$. In the present section, for the case of the radiation field, we avoid the inversion of the derivative with the integral operation. Regarding the commutation with the Taylor expansion, we consider that such an operation cannot be avoided as long as we want to emphasize the multipolar moments.

In conclusion, employing Jefimenko's equation in the radiation theory could bring a new insight only if the inversion of the spatial derivative and the integral operation is not allowed. We admit we were unable to find an interesting example where an inversion is not permitted, at least for generalized distributions. However, it might be possible to find such examples, and, in this case, the indispensable character of Jefimenko's equations would be obvious. Otherwise, for the regular cases, it appears as an unnecessary complication.

5. CONCLUSIONS AND DISCUSSION

In section 2, we presented the formalism for the multipolar expansions of the electric and magnetic field hiding a tensorial algebra procedure behind a vectorial one. Sections 3 and 4 were dedicated to the main purpose of this article: the analysis of different methods for calculating the radiated field with an emphasis on the novelty the use of Jefimenko's equations can bring.

From the analysis of sections 3 and 4 we can draw some conclusions on the utility of Jefimenko's equations when considering the multipolar expansion problem. As one can notice, when deriving equation (8) from (7), no matter if we work with potentials or directly with fields, the inversion of the spatial derivative with the integral on the domain $\mathcal{D}$ is mandatory. In section 4 it was proven that the multipole expansion of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ can be obtained generally and directly from Jefimenko's equations. All expanding operations are performed on the integrand, but, still under the assumption that the integration operation is distributive with respect to the Taylor series. It remains only to argue the necessity of the corresponding additional calculation effort for applying this procedure.

As an additional remark, we point out that in the same sections we tried to remind the reader that if one wants to completely describe the radiative systems, one has to include besides terms of order $1/r$, the $1/r^2$ contributions from the field expansions.

Ref. [2] is part of a paper series trying to emphasize the theoretical and practical importance of Jefimenko's equations. These equations are considered as *extraordinarily powerful and illuminating* as the authors of Ref. [13] write. We have nothing against the open enthusiasm in these papers. We neither dispute the beauty of the result regarding the calculation of the retarded fields $\boldsymbol{E}$ and $\boldsymbol{B}$ directly from Maxwell's equations, without having to introduce and handle the potentials. Maybe a series of applications based on these equations are more efficacious and physically more transparent. Although, we are circumspect concerning the axiomatic treatment of the electromagnetic theory starting from these equations (opposite to opinions from e.g. Refs. [14, 15]), but, this subject will be discussed elsewhere.

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